

ON NUMERICAL METHODS FOR INVERSE EIGENVALUE PROBLEMS BASED ON TRACE FORMULAE AND CONVERGENCE ANALYSIS

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ABSTRACT. Numerical methods for generalized inverse eigenvalue problems featuring multiple eigenvalues have garnered the interests of researchers. In the present paper, we propose a novel inversion method to solve inverse eigenvalue problems based on trace formulas and bridge the unknown coefficients and eigenvalues explicitly. This method displays a certain degree of universality, which is applicable for inverse eigenvalue problems with complex and multiple eigenvalues, including non-diagonalizable cases. Moreover, a rigorous convergence analysis is conducted for the proposed method. Numerical results confirm the theoretical prediction of quadratic convergence.

1. Introduction and problem formulation. Inverse eigenvalue problems (IEPs), which involve reconstructing a matrix from prescribed spectral data, have attracted significant attention due to their broad theoretical and practical implications, such as control theory, structural dynamics, signal and image processing, quantum mechanics, and bioinformatics [5, 7, 10, 11, 15]. These problems focus on constructing a matrix that satisfies prescribed structural constraints while simultaneously possessing the desired spectral properties from spectral data. The spectral data may include complete or partial information about eigenvalues and eigenvectors, depending on the specific application requirements. For instance, in inverse differential equations problems, the IEPs are employed to determine system parameters from spectra as demonstrated in [9]. Inverse Sturm-Liouville problems and the nuclear spectroscopy problems which require solutions through inverse eigenvalue problems with large dimension were discussed in [11]. More applications in various fields can be found in a survey paper [6] on structured inverse eigenvalue problems.

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Here, we consider a bounded linear operator A which maps $\mathbb{R}^M$ to $\mathbb{R}^{N \times N}$, i.e.,

$$A(\mathbf{c}) = A_0 + \sum_{m=1}^M c_m A_m, \quad \mathbf{c} = (c_1, c_2, \dots, c_M)^T \in \mathbb{R}^M, \quad \{A_m\}_{m=1}^M \in \mathbb{R}^{N \times N}. \quad (1.1)$$

For any coefficient $\mathbf{c} \in \mathbb{R}^M$ and matrix $A(\mathbf{c})$ in $\mathbb{R}^{N \times N}$, we use $\{\lambda_i(\mathbf{c})\}_{i=1}^N \in \mathbb{C}$ to denote the eigenvalues of the matrix $A(\mathbf{c})$. The *inverse eigenvalue problem* investigated in this paper is defined as follows. Find a vector $\mathbf{c}^* \in \mathbb{R}^M$ for a given set of N numbers of eigenvalues $\{\lambda_i^*\}_{i=1}^N$, such that

$$\lambda_i(\mathbf{c}^*) = \lambda_i^*, \quad i = 1, 2, \dots, N. \quad (1.2)$$

Our goal is to propose an efficient algorithm for solving this inverse eigenvalue problem in different cases especially for multiple eigenvalues and non-diagonalizable matrices. Obviously, solving this inverse eigenvalue problem can be easily reformulated as solving the following nonlinear equation $\mathbf{f}(\mathbf{c}) = 0$, where $\mathbf{f}: \mathbb{R}^M \rightarrow \mathbb{R}^N$ is defined by

$$\mathbf{f}(\mathbf{c}) = (\lambda_1(\mathbf{c}) - \lambda_1^*, \lambda_2(\mathbf{c}) - \lambda_2^*, \dots, \lambda_N(\mathbf{c}) - \lambda_N^*)^T. \quad (1.3)$$

A substantial body of literature addresses the conditions for the existence and uniqueness of solutions to inverse eigenvalue problems (IEPs) and their variations in various special cases. Detailed analyses of the existence conditions for solutions to problem (1.1) are presented in [7]. Under specific conditions, the uniqueness of IEPs for banded matrices and particular classes of real symmetric matrices is examined in [12] and [19].

Additionally, numerous numerical methods have been proposed to solve the above inverse eigenvalue problems. When the given eigenvalues $\{\lambda_i^*\}_{i=1}^N$ are distinct, there exists a neighborhood of $\mathbf{c}^* \in \mathbb{R}^M$ such that the nonlinear function $\mathbf{f}$ is analytic as shown in [8]. Newton-type methods have been proposed to solve the IEPs with distinct eigenvalues in [1, 3, 7, 11]. Subsequently, in order to improve the efficiency of calculation and alleviate the over-solving problem, constantly innovative works are discussed in [4, 20, 26]. On the other hand, when the given eigenvalues are not distinct, solving IEPs becomes more challenging, for instance, with multiple eigenvalues or the matrix operators can not be diagonalizable. The difficulties lie in the fact that $\mathbf{f}(\mathbf{c})$ is generally not differentiable at $\mathbf{c}^*$, and the corresponding eigenvectors of $A(\mathbf{c}^*)$ are not unique. To address these challenges, several extensions in the field of IEPs involving multiple eigenvalues have been developed. Shen et al. introduced an inexact Cayley transform method for multiple conditions exhibiting superlinear convergence in [21] and conducted convergence analyses of the inexact Newton-like method and the Ulm-like Cayley transform method for IEPs with multiple eigenvalues in [22] and [23]. Additionally, a two-step Ulm-Chebyshev-like Cayley transform method for addressing cases involving multiple eigenvalues is presented in [25]. However, these methods were mostly designed for specific circumstance and need sophisticated treatment case by case. These limitations highlight the need for a more robust and universally applicable approach to solving a variety of IEPs.

In this paper, we propose a novel algorithm based on trace formulas to address the challenges associated with IEPs. The traces of operators are frequently employed in addressing inverse problems in mathematical physics. In the investigation of inverse eigenvalue problems pertinent to quantum mechanics and covariance approximation in statistical estimation theory as discussed in [7], the traces of Hamilton matrices

and covariance matrices of random variables are essential to addressing these challenges. Furthermore, trace formulas for differential operators have been extensively utilized in the literature [2, 13, 14, 27]. Inspired by the approaches in [27] for the inversion of Sturm-Liouville operators, we propose an algorithm based on trace formulas for solving the inverse eigenvalue problem. Our method requires only the trace of the matrix $A(\mathbf{c})$, eliminating the need for detailed eigenvalue and eigenvector data. This significantly reduces the input data requirements and enhances the method's applicability to a wider range of problems. By carefully selecting orthogonal polynomials, we establish a stable and efficient inversion scheme that maps the coefficient vector $\mathbf{c}$ to the trace of the matrix $A(\mathbf{c})$.

$$\mathbf{c} \rightarrow \text{trace}(A(\mathbf{c})) := \sum_{i=1}^N \lambda_i(A(\mathbf{c})). \quad (1.4)$$

From the trace formula, this relation can be extended to a system of nonlinear maps

$$\mathbf{c} \rightarrow \left\{ \text{trace}(A^s(\mathbf{c})) \right\}_{s=1}^S := \left\{ \sum_{i=1}^N \lambda_i^s(A(\mathbf{c})) \right\}_{s=1}^S. \quad (1.5)$$

To ensure numerical stability while efficiently computing multidimensional integrals in trace formulas, we carefully select a sequence of orthogonal polynomials $\{T_s\}_{s=1}^S$ to obtain the maps

$$\mathbf{c} \rightarrow \left\{ \text{trace}(T_s(A(\mathbf{c}))) \right\}_{s=1}^S := \left\{ T_s\left(\sum_{i=1}^N \lambda_i(A(\mathbf{c}))\right) \right\}_{s=1}^S. \quad (1.6)$$

Since the eigenvalue distribution extends beyond the real axis, the selection of appropriate stabilizing polynomials, which depends on the spectrum distribution, is significant.

The remainder of this paper is organized as follows. In Section 2, we derive the trace formulas and present an inversion algorithm associated with implementation details. Section 3 provides a rigorous analysis of the algorithm's convergence. Numerical examples demonstrating the efficiency of the proposed algorithm are presented in Section 4, along with a discussion of parameter impacts. Finally, concluding remarks are presented in Section 5.

2. Inversion scheme.

2.1. Trace formulas. In this section, we propose an algorithm based on trace formulas, which is introduced by [27] for solving the inverse eigenvalue problems.

As shown in (1.2) and (1.3), the process of solving the unknown parameter $\mathbf{c}$ can be converted to solving a set of nonlinear equations $\mathbf{f}$. Hence, a novel scheme is proposed to derive a sequence of nonlinear equations based on trace formulas. We establish a relationship between coefficient vector $\mathbf{c} \in \mathbb{R}^M$ and the traces of $A(\mathbf{c}) \in \mathbb{R}^{N \times N}$ defined in (1.1) rather than directly relating $\mathbf{c}$ to the eigenvalues $\{\lambda_i(\mathbf{c})\}_{i=1}^N$.

For a bounded linear operator A , if $\{\lambda_i(\mathbf{c})\}_{i=1}^N$ are a set of eigenvalues of $A(\mathbf{c})$, with higher-order trace formulas, the trace of $A(\mathbf{c})$ and its powers satisfy

$$\left\{ \text{trace}(A^s(\mathbf{c})) \right\}_{s=1}^S = \left\{ \sum_{i=1}^N \lambda_i^s(\mathbf{c}) \right\}_{s=1}^S.$$

Thus, a more explicit relation can be obtained between the coefficient vector $\mathbf{c}$ and the eigenvalue data as

$$\mathbf{c} \rightarrow \text{trace}(A^s(\mathbf{c})).$$

By incorporating the algebraic multiplicities of the eigenvalues, an objective function $\tilde{\mathcal{F}}$ can be formulated as

$$\tilde{\mathcal{F}}(\mathbf{c}) = \begin{pmatrix} \text{trace}(A(\mathbf{c})) - \sum_{i=1}^N \lambda_i^* \\ \text{trace}(A^2(\mathbf{c})) - \sum_{i=1}^N \lambda_i^{*2} \\ \vdots \\ \text{trace}(A^S(\mathbf{c})) - \sum_{i=1}^N \lambda_i^{*S} \end{pmatrix} = 0. \quad (2.1)$$

In this case, this approach requires only the traces of $A(\mathbf{c}^*)$ rather than complete eigenvalue information. $\tilde{\mathcal{F}}$ constitutes a sequence of bounded linear operators that are Fréchet differentiable in $C(\mathbb{R}^M)$. Denote $\tilde{J}(\mathbf{c})$ as the Jacobian matrix of $\tilde{\mathcal{F}}(\mathbf{c})$ at each point $\mathbf{c}$, i.e.,

$$\tilde{J}_{s,m}(\mathbf{c}) = \frac{\partial \tilde{\mathcal{F}}_s(\mathbf{c})}{\partial c_m}, \quad s = 1 \dots S, \quad m = 1 \dots M.$$

Assume the Jacobian $\tilde{J}(\mathbf{c})$ is non-singular and its inverse remains bounded across all iterations. Though equation (2.1) bridges the unknown coefficients and traces explicitly, the assumption of $\tilde{J}(\mathbf{c})$ guarantees the iteration would not be broken down if we applied gradient based optimization method. In addition, IEPs associated with (2.1) encounter a similar level of difficulty as the inverse spectral problems studied in [27], [2], and [16]. That is, measuring the residual as $\|\text{trace}(A^s(\mathbf{c})) - \text{trace}(A^s(\mathbf{c}^*))\|_2$ can lead to severely instability due to the significant variation in the magnitudes of $\sum_{i=1}^N \lambda_i^s$ for different s , resulting in unsatisfactory solutions for the nonlinear equations. In general, solving the nonlinear equations defined in (2.1) is severely ill-posed. Therefore, more appropriate metrics are required to effectively distinguish between different coefficient vectors $\mathbf{c}$.

2.2. Shifted Legendre polynomials. To overcome the instability, we instead invert a different map using a sequence of chosen orthogonal polynomials T_s defined on $[0, 1]$. Namely, let $\{T_s\}_{s=1}^S$ be a sequence of polynomials with $T_s(0) = 0$. A new map is defined as

$$\mathcal{F} : \mathbf{c} \rightarrow \left\{ \sum_{i=1}^N T_s(\lambda_i) \right\}_{s=1}^S = \left\{ \text{trace}(T_s(A(\mathbf{c}))) \right\}_{s=1}^S. \quad (2.2)$$

Specifically, $T_s(A(\mathbf{c}))_{s=1}^S$ denotes a sequence of finite-dimensional matrices, where each trace is defined as the sum of its diagonal elements. In the case of real eigenvalues, inverse eigenvalue problems preserve the real-valued nature of the traces under orthogonal polynomial transformations, yielding the relation $\{T_s(\text{trace}(A(\mathbf{c})))\}_{s=1}^S = \{\sum_{i=1}^N T_s(\lambda_i)\}_{s=1}^S$. When eigenvalues are complex, careful selection of orthogonal polynomials is necessary to ensure the validity of subsequent computations.

In our framework, we employ shifted Legendre polynomials defined by the following recursive relations:

$$\tilde{T}_s(x) = \frac{2s-1}{s}(2x-1)\tilde{T}_{s-1}(x) - \frac{s-1}{s}\tilde{T}_{s-2}(x), \quad s \geq 2,$$

$$\tilde{T}_0(x) = 1, \quad \tilde{T}_1(x) = 2x - 1.$$

The shifted Legendre polynomials $\tilde{T}_s$ is orthogonal on $[0, 1]$, which is favorable to our needs. In order to satisfy $T_s(0) = 0$ and $\{\sum_{i=1}^N T_s(\lambda_i)\}_{s=1}^S = \{\text{trace}(T_s(A(\mathbf{c})))\}_{s=1}^S$, we denote $T_s(x) = x\tilde{T}_s(x)$, obtaining the recursive formulas

$$\begin{cases} T_1(x) = x, \\ T_2(x) = 2x^2 - x, \\ \dots, \\ T_{s+1}(x) = \frac{2s-1}{s}(2x-1)T_s(x) - \frac{s-1}{s}T_{s-1}(x). \end{cases} \quad (2.3)$$

The nonlinear equations in (2.1) can be reformulated as

$$\begin{pmatrix} \text{trace}(T_1(A(\mathbf{c}))) - \sum_{i=1}^N T_1(\lambda_i^*) \\ \text{trace}(T_2(A(\mathbf{c}))) - \sum_{i=1}^N T_2(\lambda_i^*) \\ \vdots \\ \text{trace}(T_S(A(\mathbf{c}))) - \sum_{i=1}^N T_S(\lambda_i^*) \end{pmatrix} = 0. \quad (2.4)$$

To evaluate the forward map and its Jacobian, we utilize the following recursive relations for the bounded linear operators $\{T_s\}_{s=2}^{S-1}$ as

$$T_{s+1}(A(\mathbf{c})) = \frac{2s-1}{s}(2A(\mathbf{c}) - I)T_s(A(\mathbf{c})) - \frac{s-1}{s}T_{s-1}(A(\mathbf{c})). \quad (2.5)$$

The corresponding differential functions are given by

$$\begin{aligned} \frac{\partial \text{trace}(T_{s+1}(A(\mathbf{c})))}{\partial c_m} &= \text{trace} \left(\frac{\partial T_{s+1}(A(\mathbf{c}))}{\partial c_m} \right) \\ &= \text{trace} \left(\frac{4s-2}{s} \frac{\partial A(\mathbf{c})}{\partial c_m} T_s(A(\mathbf{c})) + \frac{2s-1}{s} (2A(\mathbf{c}) - I) J_{s,m}(\mathbf{c}) - \frac{s-1}{s} J_{s-1,m}(\mathbf{c}) \right), \end{aligned} \quad (2.6)$$

where $s = 2 \dots S-1$ and the derivative of the matrix operator $A(\mathbf{c})$ with respect to the parameter c_m is given by

$$\frac{\partial A(\mathbf{c})}{\partial c_m} = A_m.$$

Consequently, the set $\{T_s\}_{s=1}^S$ of orthogonal polynomials is adopted as the basis for the algorithm. The inverse eigenvalue problem aims to invert the coefficients $\mathbf{c}$ in a finite-dimensional matrix space based on the nonlinear equations defined in (2.4) with a sequence of polynomials $\{T_s\}_{s=1}^S$.

2.3. Numerical method. In terms of numerical computation, for the use of the orthogonal polynomials $\{T_s\}_{s=1}^S$ defined in (2.3) and to mitigate the increased sensitivity caused by the significant variation of $\{\sum_{i=1}^N T_s(\lambda_i)\}_{s=1}^S$ with the highest degree S , we rescale the matrix $A(\mathbf{c})$ and develop a numerically stable computational algorithm in this part.

To obtain stable computational results, we first preprocess the eigenvalues and their spectra. Without loss of generality, if the eigenvalues are real, we assume the eigenvalues can be ordered as follows:

$$\lambda_1^* = \cdots = \lambda_k^* < \lambda_{k+1}^* < \cdots < \lambda_N^*, \quad k = 1 \dots N-1. \quad (2.7)$$

If the eigenvalues are not real, it is demonstrated that if $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix operator A then its conjugate $\bar{\lambda}$ is likewise an eigenvalue. Furthermore, λ and $\bar{\lambda}$ share the same geometric and algebraic multiplicities.

Given a sequence of $\{\lambda_i\}_{i=1}^N \in \mathbb{C}$ discussed previously, we select a positive number $U > \sup_{i=1 \dots N} |\Re(\lambda_i)|$. A uniform partition for the interval $[0, 1/U]$ is constructed,

where $0 = t_1 < t_2 < \cdots < t_L = 1/U$. For each t_l , where $l = 1, 2, \dots, L$, we rescale the matrix $A(\mathbf{c})$ and its eigenvalues $\{\lambda_i\}_{i=1}^N$. Then, with the orthogonal polynomials $\{T_s\}_{s=1}^S$, the inverse eigenvalue problem comes down to solving a system of nonlinear equations with the unknown vectors $\mathbf{c} = \{c_1, c_2, \dots, c_M\}$, i.e.,

$$\mathcal{F}(\mathbf{c}) = \begin{pmatrix} \text{trace}(T_S(t_1 A(\mathbf{c}))) - \sum_{i=1}^N T_S(t_1 \lambda_i^*) \\ \text{trace}(T_S(t_2 A(\mathbf{c}))) - \sum_{i=1}^N T_S(t_2 \lambda_i^*) \\ \vdots \\ \text{trace}(T_S(t_L A(\mathbf{c}))) - \sum_{i=1}^N T_S(t_L \lambda_i^*) \end{pmatrix} \triangleq \begin{pmatrix} \mathcal{F}_1(\mathbf{c}) \\ \mathcal{F}_2(\mathbf{c}) \\ \vdots \\ \mathcal{F}_L(\mathbf{c}) \end{pmatrix} = 0. \quad (2.8)$$

Here, S denotes the selected highest degree of the orthogonal polynomials. To guarantee that the above system has a locally unique solution, we need to choose $L \geq M$. The selection of the highest S and rescaled mesh L should be larger than the number of eigenvalues N to discriminate the information of all those eigenvalues as well. For the nonlinear system in (2.8), we define the bounded linear maps $\{T_{S+1}(t_l A(\mathbf{c}))\}_{l=1}^L$ and their Jacobian $J(\mathbf{c}) = (J_{l,m}(\mathbf{c}))_{L \times M}$ as follows:

$$T_{S+1}(t_l A(\mathbf{c})) = \frac{2S-1}{S}(2t_l A(\mathbf{c}) - I)T_S(t_l A(\mathbf{c})) - \frac{S-1}{S}T_{S-1}(t_l A(\mathbf{c})) \quad (2.9)$$

and

$$\begin{aligned} J_{l,m}^{S+1}(\mathbf{c}) &= \frac{\partial \text{trace}(T_{S+1}(t_l A(\mathbf{c})))}{\partial c_m} = \text{trace} \left(\frac{\partial T_{S+1}(t_l A(\mathbf{c}))}{\partial c_m} \right) \\ &= \text{trace} \left(\frac{(4S-2)t_l}{S} \frac{\partial A(\mathbf{c})}{\partial c_m} T_S(t_l A(\mathbf{c})) + \frac{2S-1}{S} (2t_l A(\mathbf{c}) - I) J_{l,m}^S(\mathbf{c}) \right) \\ &\quad - \text{trace} \left(\frac{S-1}{S} J_{l,m}^{S-1}(\mathbf{c}) \right), \end{aligned} \quad (2.10)$$

where

$$\frac{\partial A(\mathbf{c})}{\partial c_m} = A_m.$$

Using the least square method, we aim to calculate $\min \|\mathcal{F}(\mathbf{c})\|_2$. Throughout, this paper, $\mathcal{F}_l(\mathbf{c})$ is a set of differentiable bounded linear operators. For each chosen value of S , the Jacobian matrix $J(\mathbf{c})$ of the operator $\mathcal{F}_l$ at every singular point $\mathbf{c}$ is defined by

$$J_{l,m}(\mathbf{c}) = \frac{\partial \mathcal{F}_l(\mathbf{c})}{\partial c_m}. \quad (2.11)$$

Next, we apply line search method to invert the unknown value $\mathbf{c}$ with the update rule

$$\mathbf{c}^{k+1} = \mathbf{c}^k + \mu_k \delta_k,$$

where the step size μ_k satisfies the Armijo conditions, and the search direction δ_k is the Newton direction. Since systems of nonlinear equations often contain singular points, this could lead to instability in the solution. To mitigate this issue, we modify the Newton direction by adding a regularization term. Specifically, δ_k is approached by

$$\delta_k = (J^T(\mathbf{c}^k)J(\mathbf{c}^k) + \sigma_k I)^{-1} J(\mathbf{c}^k)^T \mathcal{F}(\mathbf{c}^k),$$

where σ_k is a regularization parameter and I is the identity matrix. A more detailed computation procedure is provided in Algorithm 1.

Algorithm 1 Iterative algorithm for inverse eigenvalue problems

- 1: Given maximum number iteration K and highest degree of polynomials S ;
 - 2: precompute real traces $tr^* = (\sum_{i=1}^N T_S(t_1 \lambda_i^*), \dots, \sum_{i=1}^N T_S(t_L \lambda_i^*))$;
 - 3: given initial guess $\mathbf{c}^0$;
 - 4: **for** $1 \leq k \leq K$ max number of iterations **do**
 - 5: **for** $1 \leq l \leq L$ **do**
 - 6: Calculate $tr = (\text{trace}(T_S(t_1 A(\mathbf{c}))), \dots, \text{trace}(T_S(t_l A(\mathbf{c}))))$;
 - 7: **for** $1 \leq m \leq M$ **do**
 - 8: Calculate $J_{l,m} = \frac{\partial \text{trace}(T_S(t_l A(\mathbf{c})))}{\partial c_m}$;
 - 9: **end for**
 - 10: **end for**
 - 11: Compute $\mu_k \delta_k$ by line search Newton-like method with Jacobi matrix $J = (J_{l,m})_{L \times M}$ and the residual $r = tr^* - tr$;
 - 12: $\mathbf{c}^k = \mathbf{c}^{k-1} + \mu_{k-1} \delta_{k-1}$.
 - 13: **end for**
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Remark 1. The given eigenvalues $\{\lambda_i^*\}_{i=1}^N$ of a finite, dimensional matrix may occur in complex conjugates pairs. Note that $\{T_s(x)\}_{s=1}^S$ are a sequence of orthogonal polynomials satisfying $T_s(0) = 0$ and the matrix operator $A(\mathbf{c}) \in \mathbb{R}^{N \times N}$. Based on works in [2] and [16], for each t_l , we have

$$\text{Im} \left(\sum_{i=1}^N T_S(t_l \lambda_i) \right) = 0,$$

under the particular choice of $N \in \mathbb{N}$ and $S \in \mathbb{N}^+$.

Remark 2. For a matrix operator $A \in \mathbb{R}^{N \times N}$, there is a one-to-one correspondence between the eigenvalues $\{\lambda_i\}_{i=1}^N \in \mathbb{C}$ and its traces $\{\text{tr}(A^s)\}_{s=1}^S$ when $S \geq N$.

Proof. Once the eigenvalues $\{\lambda_i\}_{i=1}^N \in \mathbb{C}$ are determined, the trace of the matrix operator A is uniquely determined as $\text{trace}(A^s) = \sum_{i=1}^N \lambda_i^s$, where $s = 1 \dots S$. Furthermore, when $S \geq N$ and N is the number of eigenvalues, consider the corresponding

system of nonlinear equations

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_N = \text{tr}(A), \\ \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_N^2 = \text{tr}(A^2), \\ \vdots \\ \lambda_1^S + \lambda_2^S + \cdots + \lambda_N^S = \text{tr}(A^S). \end{cases} \quad (2.12)$$

Let e_k be the k -th elementary symmetric polynomial. When $S \geq N$ and the power sums $\{\text{tr}(A^s)\}_{s=1}^S$ satisfy Newton's identities from the Cayley-Hamilton theorem, the eigenvalue polynomial $P(\lambda)$ is uniquely determined by its coefficients $e_1, \dots, e_n$ where

$$P(\lambda) = \prod_{i=1}^N (\lambda - \lambda_i) = \lambda^N - e_1 \lambda^{N-1} + e_2 \lambda^{N-2} - \cdots + (-1)^N e_N. \quad (2.13)$$

By the fundamental theorem of algebra, $P(\lambda)$ has exactly N roots $\lambda_1, \dots, \lambda_N$, where $\lambda_i \in \mathbb{C}$. Then, the eigenvalues $\{\lambda_i\}_{i=1}^N$ of A are determined uniquely by the power sums $\{\sum_{i=1}^N \lambda_i^s\}_{s=1}^S$. $\square$

Remark 3. The preceding remark demonstrates that there exists a one-to-one correspondence between a set of traces of matrix $\{A^s(\mathbf{c})\}_{s=1}^S$ and the eigenvalues $\{\lambda_i\}_{i=1}^N$ of matrix $A(\mathbf{c})$ when $S \geq N$. Consequently, when examining the uniqueness of problem (1.1) via traces, it suffices to investigate the uniqueness of IEPs via eigenvalues $\{\lambda_i\}_{i=1}^N$. For special cases when $A(\mathbf{c})$ is a banded matrix or a sparse real symmetric matrix with particular patterns, the uniqueness under the number of eigenvalues N and unknown coefficients $\{\mathbf{c}_m\}_{m=1}^M$ are extensively discussed in [7, 12, 19]. The uniqueness theory for generalized parameterized inverse eigenvalue problems remains an open area of research, presenting valuable and promising directions for further investigation.

3. Convergence analysis. In this section, we will establish the convergence theorem of the proposed method. Let the coefficient vector $\mathbf{c}^* \in \mathbb{R}^M$ be the exact solution of IEPs in (1.2). The bounded linear operator $\mathcal{F}(\mathbf{c}^*)$ and its Jacobian matrix $J(\mathbf{c}^*) = (J_{l,m}(\mathbf{c}^*))_{L \times M}$ are given by equations (2.8) and (2.11), respectively. Denote the open ball in $\mathbb{R}^M$ centered at $\mathbf{c}^* \in \mathbb{R}^M$ with radius $r > 0$ by $B(\mathbf{c}^*, r)$. Let the vector $\mathbf{c}^k \in B(\mathbf{c}^*, r)$ denote the k -th iterate generated by Algorithm 1, where $\mathbf{c}^0$ is the initial approximation. At each iteration k , the eigenvalues $\{\lambda_i(\mathbf{c}^k)\}_{i=1}^N \in \mathbb{C}$ correspond to the spectrum of the matrix $A(\mathbf{c}^k)$ defined in (1.1). Furthermore, the objective function $\mathcal{F}(\mathbf{c}^k)$ and its Jacobian matrix $J(\mathbf{c}^k) = (J_{l,m}(\mathbf{c}^k))_{L \times M}$ are generated by the nonlinear system (2.8) and Algorithm 1, respectively.

Lemma 3.1. *For a matrix $A \in \mathbb{R}^{m \times n}$, the generalized inverse of the matrix A introduced by Moore and Penrose in [17] and [18] is denoted as $A^\dagger$. When $m \geq n$ and matrix A is full rank, the Moore-Penrose inverse is given by $A^\dagger = (A^T A)^{-1} A^T$. If A is square and non-singular, the Moore-Penrose generalized inverse coincides with the usual inverse A^{-1} .*

Lemma 3.2. [24, Lemma 5.1] *$A, E \in \mathbb{R}^{m \times n}$, $B = A + E$. If $\|A^\dagger\| \|E\| \leq 1$ and $\text{rank}(B) = \text{rank}(A)$, then*

$$\|B^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger E\|}.$$

If $\text{rank}(A) = \text{rank}(B) = \min\{m, n\}$, then

$$\|B^\dagger - A^\dagger\| \leq \frac{\sqrt{2}\|A^\dagger\|^2\|E\|}{1 - \|A^\dagger E\|}.$$

Definition 3.3. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Denote the space of bounded linear operators from X to Y by $L_c(X, Y)$. Let $A \subseteq X$ be an open set and $\mathcal{F} : A \rightarrow Y$ be a bounded linear map of class C^1 . That is, $\mathcal{F}$ is Fréchet differentiable at every $x \in A$ and the Fréchet derivative $\mathcal{F}'(x) \in L_c(X, Y)$ is denoted as

$$\lim_{h \rightarrow 0} \frac{\|\mathcal{F}(x+h) - \mathcal{F}(x) - \mathcal{F}'(x)h\|_Y}{\|h\|_X} = 0.$$

Consider the bounded linear operator $\mathcal{F} : \mathbb{R}^M \rightarrow \mathbb{C}$ and its Fréchet derivative $\mathcal{F}'$, which is generated by (2.8) in $C(\mathbb{R}^M)$. In particular, the Fréchet derivative $\mathcal{F}'$ at differentiable points $\mathbf{c} \in B(\mathbf{c}^*, r)$ is defined as the Jacobian

$$J(\mathbf{c}) = (J_{l,m}(\mathbf{c}))_{L \times M},$$

where $J_{l,m} : \mathbb{R}^M \rightarrow \mathbb{C}$ are bounded linear operators defined in (2.11).

Lemma 3.4. Let $\mathbf{c}^* \in \mathbb{R}^M$ be a point where the matrix $A(\mathbf{c}^*)$ has eigenvalues satisfying relations in (1.2). Assuming that the Jacobian matrix $J(\mathbf{c}^*)$ is non-singular, there exist positive constants r and μ such that for all $\mathbf{c} \in B(\mathbf{c}^*, r)$,

$$\sup_{\mathbf{c} \in B(\mathbf{c}^*, r)} \|J(\mathbf{c})^{-1}\| \leq \frac{1}{\mu}.$$

Lemma 3.5. Suppose $\mathbf{c}^* \in \mathbb{R}^M$ is a solution to the inverse eigenvalue problem (1.1). There exists a neighborhood $B(\mathbf{c}^*, r)$ of $\mathbf{c}^*$ and positive constants θ and γ such that for all $\mathbf{c} \in B(\mathbf{c}^*, r)$, the nonlinear function $\mathcal{F}(\mathbf{c})$ and its Fréchet derivative $J(\mathbf{c})$ are Lipschitz continuous with constants θ and γ , respectively. Specifically,

$$\|\mathcal{F}(\mathbf{c}^k) - \mathcal{F}(\mathbf{c}^*)\| \leq \theta \|\mathbf{c}^k - \mathbf{c}^*\|, \quad (3.1)$$

and

$$\|J(\mathbf{c}^k) - J(\mathbf{c}^*)\| \leq \gamma \|\mathbf{c}^k - \mathbf{c}^*\|, \quad (3.2)$$

where $\|\cdot\|$ denotes the L_2 norm.

Proof. Based on equation (2.8) and Lemma 3.3, $\mathcal{F}$ is a bounded linear operator, and its Fréchet derivative operator J satisfies $\max \|J(\mathbf{c})\| \leq \theta$, where $\mathbf{c} \in B(\mathbf{c}^*, r)$. Therefore, we obtain

$$\|\mathcal{F}(\mathbf{c}^k) - \mathcal{F}(\mathbf{c}^*)\| \leq \int_0^1 \|J(\mathbf{c}^k + t(\mathbf{c}^* - \mathbf{c}^k))\| dt \|\mathbf{c}^k - \mathbf{c}^*\| \leq \theta \|\mathbf{c}^k - \mathbf{c}^*\|.$$

Thus, inequality (3.1) holds based on the boundedness properties of the operator J . Specifically,

$$\|\mathcal{F}(\mathbf{c}^k) - \mathcal{F}(\mathbf{c}^*)\| = \|\text{trace}(T_S(t_l A(\mathbf{c}))) - \text{trace}(T_S(t_l A(\mathbf{c}^*)))\|.$$

The following inequality holds for the Jacobian matrix $J(\mathbf{c}^k) = (J_{l,m}(\mathbf{c}^k))_{L \times M}$ defined in (2.10), where

$$\begin{aligned}
|J_{l,m}^{S+1}(\mathbf{c}^k) - J_{l,m}^{S+1}(\mathbf{c}^*)| &\leq \frac{4S-2}{S} |\text{trace}(A_m T_S(t_l A(\mathbf{c}^k)) - A_m T_S(t_l A(\mathbf{c}^*)))| \\
&+ \frac{2S-1}{S} |\text{trace}(t_l A(\mathbf{c}^k) J_{l,m}^S(\mathbf{c}^k) - t_l A(\mathbf{c}^*) J_{l,m}^S(\mathbf{c}^*) + J_{l,m}^S(\mathbf{c}^*) - J_{l,m}^S(\mathbf{c}^k))| \\
&+ \frac{S-1}{S} |\text{trace}(J_{l,m}^{S-1}(\mathbf{c}^k) - J_{l,m}^{S-1}(\mathbf{c}^*))|.
\end{aligned}$$

$\{A_m\}_{m=1}^M$ is a set of finite dimensional constant matrices and $\mathcal{F}(\mathbf{c}^k)$ satisfies the Lipschitz continuity property in (3.1). Consequently, the operator $\text{trace}(A_m T_S(t_l A(\mathbf{c}^k)))$ is continuous at $\mathbf{c}^k$ for each iteration and any chosen S . Then, there exists a constant $\gamma_1 > 0$, such that for any $\mathbf{c}^k \in B(\mathbf{c}^*, r)$, we have

$$|\text{trace}(A_m T_S(t_l A(\mathbf{c}^k)) - \text{trace}(A_m T_S(t_l A(\mathbf{c}^*)))| \leq \gamma_1 |\mathbf{c}^k - \mathbf{c}^*|. \quad (3.3)$$

According to the shifted Legendre polynomials and their recursive formula as defined in (2.3), when $S = 1$, it follows that

$$|\text{trace}(J_{l,m}^1(\mathbf{c}^k) - J_{l,m}^1(\mathbf{c}^*))| = 0. \quad (3.4)$$

When $S = 2$, there exists $\gamma_2 > 0$ such that

$$|\text{trace}(J_{l,m}^2(\mathbf{c}^k) - J_{l,m}^2(\mathbf{c}^*))| = 4 |\text{trace}(t_l A(\mathbf{c}^k) - \text{trace}(t_l A(\mathbf{c}^*)))| \leq \gamma_2 |\mathbf{c}^k - \mathbf{c}^*|. \quad (3.5)$$

For the chosen $S \geq 3$, we have

$$\begin{aligned}
&|\text{trace}(A(\mathbf{c}^k) J_{l,m}^S(\mathbf{c}^k) - A(\mathbf{c}^*) J_{l,m}^S(\mathbf{c}^*))| \\
&\leq |\text{trace}((A(\mathbf{c}^k) - A(\mathbf{c}^*)) J_{l,m}^S(\mathbf{c}^k))| + |\text{trace}(A(\mathbf{c}^*) (J_{l,m}^S(\mathbf{c}^k) - J_{l,m}^S(\mathbf{c}^*)))|.
\end{aligned}$$

For a fixed coefficient vector $\mathbf{c}$ and chosen S , the operator A and the Jacobian matrix J are bounded finite-dimensional matrices. Then, there exists a positive constant $\gamma_3 > 0$ such that the $\text{trace}(A(\mathbf{c}) J_{l,m}^S(\mathbf{c}))$ satisfies the Lipschitz continuity property where

$$|\text{trace}(A(\mathbf{c}^k) J_{l,m}^S(\mathbf{c}^k) - A(\mathbf{c}^*) J_{l,m}^S(\mathbf{c}^*))| \leq \gamma_3 |\mathbf{c}^k - \mathbf{c}^*|. \quad (3.6)$$

Combining these results with inequalities (3.3) and (3.6), we have

$$\begin{aligned}
|J_{l,m}^{S+1}(\mathbf{c}^k) - J_{l,m}^{S+1}(\mathbf{c}^*)| &\leq \frac{(4S-2)\gamma_1}{S} |\mathbf{c}^k - \mathbf{c}^*| + \frac{(2S-1)\gamma_3}{S} |\mathbf{c}^k - \mathbf{c}^*| \\
&+ \frac{2S-1}{S} |\text{trace}(J_{l,m}^S(\mathbf{c}^k) - J_{l,m}^S(\mathbf{c}^*))| \\
&+ \frac{S-1}{S} |\text{trace}(J_{l,m}^{S-1}(\mathbf{c}^k) - J_{l,m}^{S-1}(\mathbf{c}^*))|.
\end{aligned}$$

For the chosen the highest degree S , by iteration of the recursive Jacobian formula, there exists a positive constant γ such that every element in the Jacobian matrix $J = (J_{l,m})_{L \times M}$ is Lipschitz continuous based on the results of the initial equations in (3.4) and (3.5), i.e.,

$$|J_{l,m}(\mathbf{c}^k) - J_{l,m}(\mathbf{c}^*)| \leq \frac{\gamma}{\sqrt{LM}} \|\mathbf{c}^k - \mathbf{c}^*\|. \quad (3.7)$$

Based on (3.7) above, we conclude that the Jacobian matrix operator $J(\mathbf{c})$ is Lipschitz continuous, where

$$\|J(\mathbf{c}^k) - J(\mathbf{c}^*)\| \leq \gamma \|\mathbf{c}^k - \mathbf{c}^*\|. \quad (3.8)$$

Thus, the proof is completed. $\square$

Based on Lemma 3.2, Lemma 3.4, and Lemma 3.5 above, the bounded linear operator $\mathcal{F}$ in the nonlinear system (2.8) has a nonsingular Jacobian matrix $J(\mathbf{c})$ for $\forall \mathbf{c} \in B(\mathbf{c}^*, r)$, where $\mathcal{F}$ and J are Lipschitz continuous with positive constants θ and γ . We now establish the local convergence theorem of Algorithm 1.

Theorem 3.6. $\{\mathbf{c}^k\}_{k=1}^K \in B(\mathbf{c}^*, r)$ and stepsize $\{\mu^k\}_{k=1}^K$ where $\max_{i=1 \dots N} |\mu^k| \leq \zeta_1$ are sequences generated by Algorithm 1. θ and γ are the Lipschitz constants defined in Lemma 3.5. Let $r > 0$ satisfy

$$0 < \frac{\alpha\gamma\zeta_1}{2(1-\alpha r\gamma)} + \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1 + \alpha\theta(1+\zeta_1)}{1-\alpha\gamma r} < 1. \quad (3.9)$$

Then, Algorithm 1 is convergent for all vectors $\mathbf{c}^k \in B(\mathbf{c}^*, r)$, and the iteration sequence $\{\mathbf{c}^k\}_{k=1}^K$ converges to $\mathbf{c}^*$ as

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \frac{\alpha\gamma\zeta_1}{2(1-\alpha r\gamma)} \|\mathbf{c}^k - \mathbf{c}^*\|^2 + \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1 + \alpha\theta(1+\zeta_1)}{1-\alpha\gamma r} \|\mathbf{c}^k - \mathbf{c}^*\|.$$

Here,

$$\epsilon = \|\mathcal{F}(\mathbf{c}^*)\|, \quad \alpha = \|(J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1}J^T(\mathbf{c}^*)\|,$$

and $\|\cdot\|$ represents the L_2 norm in $\mathbb{R}^M$. In particular, if $\mathcal{F}(\mathbf{c}^*) = 0$ and the stepsize $\mu^k \equiv 1$, the sequence $\{\mathbf{c}^k\}_{k=1}^K$ are quadratic convergent to $\mathbf{c}^*$ satisfying

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \frac{\alpha\gamma}{2(1-\alpha\|\mathbf{c}^0 - \mathbf{c}^*\|\gamma)} \|\mathbf{c}^k - \mathbf{c}^*\|^2.$$

Proof. For any coefficient vector $\mathbf{c}^k \in B(\mathbf{c}^*, r)$ where r satisfies (3.9), the nonsingular Jacobian matrix $J(\mathbf{c}^k)$ satisfies Lipschitz continuity as established in Lemma 3.4 and Lemma 3.5. Consequently, we obtain

$$\|(J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1}J^T(\mathbf{c}^*)\| \|J^T(\mathbf{c}^k) - J^T(\mathbf{c}^*)\| \leq \alpha\gamma r < 1. \quad (3.10)$$

Applying Lemma 3.1 and Lemma 3.2, we derive

$$\|(J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1}J^T(\mathbf{c}^k)\| \leq \frac{\alpha}{1-\alpha\gamma\|\mathbf{c}^k - \mathbf{c}^*\|}, \quad (3.11)$$

and

$$\|(J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1}J^T(\mathbf{c}^k) - (J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1}J^T(\mathbf{c}^*)\| \leq \frac{\sqrt{2}\alpha^2\gamma}{1-\alpha\gamma\|\mathbf{c}^k - \mathbf{c}^*\|} \|\mathbf{c}^k - \mathbf{c}^*\|. \quad (3.12)$$

According to Algorithm 1, we have

$$\begin{aligned} \mathbf{c}^{k+1} - \mathbf{c}^* &= \mathbf{c}^k - \mu_k (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) \mathcal{F}(\mathbf{c}^k) - \mathbf{c}^* \\ &= \mathbf{c}^k - \mathbf{c}^* - \mu_k (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) (\mathcal{F}(\mathbf{c}^k) - \mathcal{F}(\mathbf{c}^*)) \\ &\quad + (\mu_k (J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1} J^T(\mathbf{c}^*) - \mu_k (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k)) \mathcal{F}(\mathbf{c}^*). \end{aligned}$$

Subsequently, there is

$$\begin{aligned}
& \|\mathbf{c}^{k+1} - \mathbf{c}^*\| \\
& \leq \| (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) \| \left\| \int_0^1 (J(\mathbf{c}^k) - \mu_k J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))) (\mathbf{c}^k - \mathbf{c}^*) dt \right\| \\
& \quad + \|\mu_k (J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1} J^T(\mathbf{c}^*) - \mu_k (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k)\| \|\mathcal{F}(\mathbf{c}^*)\| \\
& \leq \| (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) \| \|\mathbf{c}^k - \mathbf{c}^*\| \int_0^1 \|J(\mathbf{c}^k) - \mu_k J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))\| dt \\
& \quad + \|\mu_k (J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1} J^T(\mathbf{c}^*) - \mu_k (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k)\| \|\mathcal{F}(\mathbf{c}^*)\|
\end{aligned}$$

If the step size $\mu_k \equiv 1$, by Lemma 3.5, we obtain

$$\int_0^1 \|J(\mathbf{c}^k) - J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))\| dt \leq \gamma \int_0^1 \|t(\mathbf{c}^k - \mathbf{c}^*)\| dt = \frac{\gamma}{2} \|\mathbf{c}^k - \mathbf{c}^*\|. \quad (3.13)$$

Thus, combining inequalities (3.11) and (3.13) gives

$$\begin{aligned}
& \| (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) \| \|\mathbf{c}^k - \mathbf{c}^*\| \int_0^1 \|J(\mathbf{c}^k) - \mu_k J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))\| dt \\
& \leq \frac{\alpha\gamma}{2(1 - \alpha\|\mathbf{c}^k - \mathbf{c}^*\|\gamma)} \|\mathbf{c}^k - \mathbf{c}^*\|^2 \leq \frac{\alpha\gamma}{2(1 - \alpha\|\mathbf{c}^0 - \mathbf{c}^*\|\gamma)} \|\mathbf{c}^k - \mathbf{c}^*\|^2.
\end{aligned} \quad (3.14)$$

If the step size is chosen to be bounded with $\mu_k \leq \zeta_1$, according to Lemma 3.5, it turns out that

$$\begin{aligned}
& \int_0^1 \|J(\mathbf{c}^k) - \mu_k J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))\| dt \\
& \leq \|J(\mathbf{c}^k)\| + \zeta_1 \int_0^1 (\|J(\mathbf{c}^k)\| + \gamma t \|\mathbf{c}^k - \mathbf{c}^*\|) dt \\
& \leq (1 + \zeta_1)\theta + \frac{\gamma\zeta_1}{2} \|\mathbf{c}^k - \mathbf{c}^*\|.
\end{aligned} \quad (3.15)$$

Then, with inequalities (3.11) and (3.15), we obtain

$$\begin{aligned}
& \| (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k) \| \|\mathbf{c}^k - \mathbf{c}^*\| \int_0^1 \|J(\mathbf{c}^k) - \mu_k J(\mathbf{c}^* + t(\mathbf{c}^k - \mathbf{c}^*))\| dt \\
& \leq \frac{\alpha\theta(1 + \zeta_1)}{1 - \alpha\gamma\|\mathbf{c}^0 - \mathbf{c}^*\|} \|\mathbf{c}^k - \mathbf{c}^*\| + \frac{\alpha\gamma\zeta_1}{2(1 - \alpha\|\mathbf{c}^0 - \mathbf{c}^*\|\gamma)} \|\mathbf{c}^k - \mathbf{c}^*\|^2.
\end{aligned} \quad (3.16)$$

Furthermore, according to inequality (3.12) and the assumption $\|\mathcal{F}(\mathbf{c}^*)\| = \epsilon$, we can obtain

$$\begin{aligned}
& \|\mu_k (J^T(\mathbf{c}^*)J(\mathbf{c}^*))^{-1} J^T(\mathbf{c}^*) - (J^T(\mathbf{c}^k)J(\mathbf{c}^k))^{-1} J^T(\mathbf{c}^k)\| \|\mathcal{F}(\mathbf{c}^*)\| \\
& \leq \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1}{1 - \alpha\gamma\|\mathbf{c}^k - \mathbf{c}^*\|} \|\mathbf{c}^k - \mathbf{c}^*\| \leq \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1}{1 - \alpha\gamma\|\mathbf{c}^0 - \mathbf{c}^*\|} \|\mathbf{c}^k - \mathbf{c}^*\|.
\end{aligned} \quad (3.17)$$

Finally, combining inequalities (3.16) and (3.17) yields

$$\begin{aligned}\|\mathbf{c}^{k+1} - \mathbf{c}^*\| &\leq \frac{\alpha\gamma\zeta_1}{2(1 - \alpha\|\mathbf{c}^0 - \mathbf{c}^*\|\gamma)}\|\mathbf{c}^k - \mathbf{c}^*\|^2 + \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1 + \alpha\theta(1 + \zeta_1)}{1 - \alpha\gamma\|\mathbf{c}^0 - \mathbf{c}^*\|}\|\mathbf{c}^k - \mathbf{c}^*\| \\ &\leq \frac{\alpha\gamma\zeta_1}{2(1 - \alpha r\gamma)}\|\mathbf{c}^k - \mathbf{c}^*\|^2 + \frac{\sqrt{2}\alpha^2\gamma\epsilon\zeta_1 + \alpha\theta(1 + \zeta_1)}{1 - \alpha\gamma r}\|\mathbf{c}^k - \mathbf{c}^*\|.\end{aligned}$$

When $\|\mathcal{F}(\mathbf{c}^*)\| = 0$ and the stepsize is fixed at $\mu_k \equiv 1$, according to (3.14), the unknown coefficients $\{\mathbf{c}^k\}$ are quadratic convergent to $\mathbf{c}^*$ and

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \frac{\alpha\gamma}{2(1 - \alpha\|\mathbf{c}^0 - \mathbf{c}^*\|\gamma)}\|\mathbf{c}^k - \mathbf{c}^*\|^2.$$

Thus, the proof is complete. $\square$

Remark 4. Suppose that $\mathbf{c}^* \in \mathbb{R}^M$ is an exact solution to the inverse eigenvalue problem in (1.1) and the initial guess $\mathbf{c}^0 \in B(\mathbf{c}^*, r)$, where r satisfies inequality (3.9). Theorem 3.6 shows that Algorithm 1 is locally quadratic convergent if the residual $\mathcal{F}(\mathbf{c}^*) = 0$. In cases where $\mathcal{F}(\mathbf{c}^*) > 0$, the rate of local convergence depends on the magnitude of the residuals. Furthermore, the line search procedure used to determine the step size at each iteration would influence the convergence rate as demonstrated in numerical experiments.

4. Numerical simulations. In this section, five numerical examples are presented to verify the validity and effectiveness of Algorithm 1 for solving the inverse eigenvalue problems. The parameters used in the examples are displayed in Table 1. The impact of these parameters are analyzed in different examples. To ensure that the nonlinear system (2.8) has a unique solution, we set $M \leq L$ for the numerical reconstruction. Additionally, the stopping tolerance for the iterations is chosen as $tol = 10^{-10}$. Since the Algorithm 1 converges locally, we select the initial guess $\mathbf{c}^0$ within a neighborhood of $\mathbf{c}^*$ satisfying $\mathbf{c}^0 \in B(\mathbf{c}^*, r)$ in Theorem 3.6 for illustrative purposes.

TABLE 1. Parameters for Algorithm 1

notation	parameter
S	the highest degree of chosen polynomials T_S
N	number of measured eigenvalues
M	size of coefficient vector $\mathbf{c}$
L	number of unknown nonlinear equations
K	maximum number of iterations

It is noted that the selection of the highest degree S needs to be greater than the number of measured eigenvalues N and the number of coefficients M . In experiments, if S is too much larger, it will also cause an over-fitting problem for its high order. Therefore, the selection of S is also quite important in practical experiments.

Example 4.1. The first example is concerned with the real symmetric Toeplitz matrices

$$A_1 = I, A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \dots, A_M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

We consider the matrix sequence $\{A_m\}_{m=1}^M \in \mathbb{R}^{N \times N}$ and vary the problem size by setting $M = 10, 50, 100$, where $M = N$ in Example 4.1. In numerical calculation, there exist integers k such that the stopping criterion $\|\mathcal{F}(\mathbf{c}^{k+1}) - \mathcal{F}(\mathbf{c}^k)\| < 10^{-10}$. The initial guess $\mathbf{c}^0$ satisfies the condition $\|\mathbf{c}^* - \mathbf{c}^0\|_2 \leq r$, where the convergence radius r satisfies the inequality in (3.9). In each case, $\mathbf{c}^*$ is generated by choosing randomly between 0 and 10. Given the constant coefficient vector $\mathbf{c}^*$, the initial guess $\mathbf{c}^0$ is generated according to $\mathbf{c}^0 = \mathbf{c}^* + \delta \cdot \text{randn}(M)$, where $\delta = 0.1$ is the noise level. Table 2 summarizes the averaged values of $\text{res}^k = \|\mathbf{c} - \mathbf{c}^*\|_2$ and the averaged total numbers of the iterations under different M . In the calculation, the

TABLE 2. Convergence performance of Example 4.1 with different problem sizes M .

Iterations	$M = 10$		$M = 50$		$M = 100$	
	res^k	$\frac{\text{res}^{k+1}}{(\text{res}^k)^2}$	res^k	$\frac{\text{res}^{k+1}}{(\text{res}^k)^2}$	res^k	$\frac{\text{res}^{k+1}}{(\text{res}^k)^2}$
2	1.1396	0.12	1.2793	0.49	2.4636	0.09
3	0.1515	0.72	0.8065	0.34	0.5547	0.40
4	0.0162	6.95	0.2208	0.10	0.1245	1.79
5	1.7340e-03	7.35	4.9817e-03	2.10	0.0279	2.18
6	2.1265e-05	7.90	5.2160e-05	5.97	0.0017	6.97
7	3.5752e-09	—	1.6267e-08	—	2.0757e-05	—

selection of the highest equation degree S is crucial. To ensure that the traces of polynomials can discriminate the information of all eigenvalues of the matrix $A(\mathbf{c})$ and to avoid over-fitting, we select $S = 2M$ in the experiments. The results are shown in Table 1.

Example 4.2. Consider a set of Hankel matrices

$$A_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 1 \\ 0 & \cdots & 1 & 1 \end{pmatrix}, \dots, A_M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \ddots & 1 & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix}.$$

The Hankel matrices in Example 4.2 are a set of matrices in which the elements on each sub-diagonal are equal. We consider three cases $M = N = 10, 50, 100$ as well in this experiment. The initial guess $\mathbf{c}^0$ is generated randomly via disturbing the corresponding element of $\mathbf{c}^*$ by 10^{-2} following the approach adopted in Example 4.1. For the case $S = 2M$, computational stability is ensured. We perform the inversion calculations on ten different sets of coefficients, and the final average convergence results are shown in Table 3. Again, a quadratic convergence rate is achieved when the cost function $\mathcal{F}(\mathbf{c}^*)$ approaches zero.

TABLE 3. Convergence performance of Example 4.2 with different problem sizes M .

Iterations	$M = 10$		$M = 50$		$M = 100$	
	res^k	$\frac{res^{k+1}}{(res^k)^2}$	res^k	$\frac{res^{k+1}}{(res^k)^2}$	res^k	$\frac{res^{k+1}}{(res^k)^2}$
2	1.0724	0.13	1.2793	0.31	2.1601	0.05
3	0.1471	0.63	0.5022	0.16	0.1296	1.65
4	0.0137	0.68	0.0412	2.00	0.0277	9.51
5	1.2894e-03	7.75	0.0034	7.91	7.3030e-03	9.60
6	1.1290e-05	8.91	9.1510e-05	7.96	5.1218e-04	5.08
7	1.1361e-09	—	6.6734e-08	—	1.3344e-06	—

Example 4.3. In this example, we consider the matrix with multiple eigenvalues. Let the real symmetric matrix $B = (b_{ij})_{8 \times 8} = I + VV^T$, where

$$V = \begin{pmatrix} 1 & -1 & -3 & -5 & -6 \\ 1 & 1 & -2 & -5 & -10 \\ 1 & -1 & -1 & 5 & 18 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & -1 \\ 2.5 & 0.2 & 0.3 & 0.5 & 0.6 \\ 2 & -0.2 & 0.3 & 0.5 & 0.8 \end{pmatrix}.$$

Define a set of matrices $\{A_m\}_{m=1}^M$ from the matrix B as follows:

$$A_m = b_{mm}e_me_m^T + \sum_{j=1}^{m-1} (b_{mj}e_me_j^T + b_{mj}e_je_m^T),$$

where $m = 1, 2, \dots, 8$ and e_m is the m th column of the identity matrix I . Then, we compute the eigenvalues $\{\lambda_i^*\}_{i=1}^N$ of $A(\mathbf{c}^*)$ as the prescribed eigenvalues, where

$$\lambda^* = (1, 1, 1, 2.9416, 7.1769, 17.8493, 38.6361, 531.406)^T.$$

We test ten cases with different initial guesses $\mathbf{c}^0$ which are generated by perturbing each entry of $\mathbf{c}^* = \{1, 1, 1, 1, 1, 1, 1, 1\}$ randomly, and choose the highest degree of orthogonal polynomials $S = 16$. The final average convergence results are shown in Table 4.

TABLE 4. Convergence performance of Example 4.3 of Algorithm 1.

Iterations	$res^k = \ \mathbf{c}^k - \mathbf{c}^*\ _2$	$\frac{res^{k+1}}{(res^k)^2}$
3	0.3738	1.15
4	0.1611	1.73
5	0.0452	2.69
6	0.0055	3.34
7	1.0112e-04	8.74
8	8.9406e-08	—

Example 4.4. In this example, the matrix A is designed such that its spectrum consists of complex eigenvalues. Let $B = (b_{ij})_{6 \times 6} \in \mathbb{R}^{6 \times 6}$, where

$$B = \begin{pmatrix} 7 & 6 & 18 & 14 & 5 & 1 \\ 3 & 4 & 11 & 6 & 14 & 13 \\ 13 & 6 & 19 & 14 & 17 & 8 \\ 16 & 9 & 13 & 14 & 7 & 19 \\ 9 & 11 & 20 & 2 & 16 & 1 \\ 2 & 10 & 5 & 6 & 14 & 10 \end{pmatrix}.$$

Define a set of matrices $\{A_m\}_{m=1}^M$ from matrix B as follows:

$$A_m = b_{mm}e_me_m^T + \sum_{j=1}^{m-1} (b_{mj}e_me_j^T + b_{jm}e_je_m^T),$$

where $m = 1, 2, \dots, 6$ and e_m is the m th column of the identity matrix I . The matrix A is constructed as a linear combination $A(\mathbf{c}) = \sum_{m=1}^6 c_m A_m$ with the coefficient vector $\mathbf{c}$. Given the ground truth $\mathbf{c}^* = \{1, 1, 1, 1, 1, 1\}$, the eigenvalues of $A(\mathbf{c}^*)$ are computed as follows:

$$\lambda^* = \{62.5956, 12.9218 \pm 7.1943i, -7.8505, -5.2944 \pm 3.6287i\}.$$

We choose $S = 12$ and precompute the initial guess $\mathbf{c}^0 = \mathbf{c}^* + \delta \cdot \text{randn}(6)$, where $\delta = 0.1$. Similarly, considering ten different sets of initial guesses $\mathbf{c}^0$, we display the average errors and convergence results of Example 4.4 in Table 5.

TABLE 5. Convergence performance on Example 4.4 of Algorithm 1.

Iterations	$res^k = \ \mathbf{c}^k - \mathbf{c}^*\ $	$\frac{res^{k+1}}{(res^k)^2}$
3	1.7694	0.10
4	0.3157	0.93
5	0.0933	2.57
6	0.0224	3.81
7	1.9104e-03	7.69
8	2.8091e-05	5.65
9	4.4602e-09	—

Example 4.5. Algorithm 1 also works for those matrices which cannot be diagonalized. In this example, we consider an inverse eigenvalue problem for non-diagonalizable matrices with multiple eigenvalues. Let $B = (b_{ij})_{N \times N} \in \mathbb{R}^{N \times N}$ with $N = 6$, where

$$B = \begin{pmatrix} 1 & 1 & 3 & 4 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

Define the matrices $\{A_m\}_{m=1}^M$ based on B as follows:

$$A_m = b_{mm}e_me_m^T + \sum_{j=1}^{m-1} (b_{mj}e_me_j^T + b_{jm}e_je_m^T),$$

where $m = 1, 2, \dots, 6$ and e_m is the m th column of the identity matrix I . Then, we construct matrix A as

$$A = \sum_{m=1}^M c_m A_m,$$

where $M = N = 6$. The eigenvalues of matrix $A(\mathbf{c}^*)$ are computed as follows:

$$\lambda^* = \{1, 1, 4, 3, 2, 7\}.$$

Given the ground truth $\mathbf{c}^* = \{1, 1, 1, 1, 1, 1\}$ and the initial guess $\mathbf{c}^0$ generated by perturbing each component of the exact solution, the inversion result is illustrated in Table 6. For a non-diagonalizable matrix A of the same form as in Example 4.5,

TABLE 6. Convergence performance on Example 4.5 of Algorithm 1.

Iterations	$res^k = \ \mathbf{c}^k - \mathbf{c}^*\ _2$	$\frac{res^{k+1}}{(res^k)^2}$
2	0.9286	0.16
3	0.1396	1.59
4	0.0310	3.02
5	0.0031	4.94
6	4.7505e-05	7.32
7	1.6524e-08	–

as long as the trace of the matrix is known, the coefficients can also be inverted through Algorithm 1.

5. Conclusions. In this paper, we propose an efficient algorithm based on the trace formula to solve the inverse eigenvalue problem involving complex and multiple eigenvalues. Rigorous analysis has been conducted and shows that the algorithm is locally quadratic convergent. Numerical examples demonstrate that the theoretical prediction of convergence rate can be achieved for different settings. Moreover, this algorithm shows a certain degree of universality, effectively solving inverse eigenvalue problems with complex and multiple eigenvalues, including non-diagonalizable matrices.

REFERENCES

- [1] Z. Bai, R. H. Chan and B. Morini, An inexact Cayley transform method for inverse eigenvalue problems, *Inverse Problems*, **20**(2004), 1675-1689.
- [2] G. Bao, X. Xu and J. Zhai, Inverse spectral problem for a damped wave operator, *SIAM Journal on Applied Mathematics*, **81** (2021), 1799-1820.
- [3] R.-H. Chan, H. Chung and S. Xu, The inexact Newton-like method for inverse eigenvalue problem, *BIT Numerical Mathematics*, **43** (2003), 7-20.
- [4] X. Chen, C. Wen and H. Sun, Two-step Newton-type methods for solving inverse eigenvalue problems, *Numerical Linear Algebra with Applications*, **25** (2018), 14 pp.
- [5] M.-T. Chu, Inverse eigenvalue problems, *SIAM Review*, **40**(1998), 1-39.
- [6] M.-T. Chu and G.-H. Golub, Structured inverse eigenvalue problems, *Acta Numerica*, **11** (2002), 1-71.
- [7] M. T. Chu and G. H. Golub, *Inverse Eigenvalue Problems: Theory, Algorithms and Applications*, Numer. Math. Sci. Comput. Oxford University Press, New York, 2005.
- [8] H. Dai and P. Lancaster, Newton's Method for a generalized inverse eigenvalue Problem, *Numer. Linear Algebra Appl.*, **4** (1997), 1-21.

- [9] A. C. Downing Jr and A. S. Householder, [Some inverse characteristic value problems](#), *Journal of the ACM*, **3** (1956), 203-207.
- [10] S. Elhay and Y. M. Ram, An affine inverse eigenvalue problem, *Inverse Problems*, **18** (2002), 455-466.
- [11] S. Friedland, J. Nocedal and M. L. Overton, [The formulation and analysis of numerical methods for inverse eigenvalue problems](#), *SIAM Journal on Numerical Analysis*, **24** (1987), 634-667.
- [12] G. M. L. Gladwell, *Inverse Problems in Vibration*, Monographs Textbooks Mech. Solids Fluids: Mech. Dynam. Systems, 9 Martinus Nijhoff Publishers, Dordrecht, 1986.
- [13] X. Jiang and X. Xu, An inverse spectral problem for a fourth-order Sturm-Liouville operator based on trace formulae, *Applied Mathematics Letters*, **111** (2021), 9 pp.
- [14] X. Jiang, X. Xu and X. Li, Numerical algorithms for inverse Sturm-Liouville problems, *Numerical Algorithms*, **89** (2022), 1287-1309.
- [15] K. T. Joseph, *Inverse Eigenvalue Problem in Structural Design*, AIAA journal, 1992.
- [16] X. Li, X. Jiang and X. Xu, Numerical algorithms for recovering a fractional Sturm-Liouville operator based on trace formulae, *Numerical Algorithms*, **99** (2024), 1623-1648.
- [17] E. H. Moore, *On the Reciprocal of the General Algebraic Matrix*, Bulletin of the American mathematical society, 1920.
- [18] R. Penrose, [A generalized inverse for matrices](#), *Proc. Cambridge Philos. Soc.*, **51** (1955), 406-413.
- [19] J. Peng, X. Hu and L. Zhang, Two inverse eigenvalue problems for a special kind of matrices, *Linear Algebra and Its Applications*, **416** (2006), 336-347.
- [20] W. Shen, C. Li and X. Jin, A Ulm-like method for inverse eigenvalue problems, *Applied Numerical Mathematics*, **61**(2011), 356-367.
- [21] W. Shen, C. Li and X. Jin, An inexact Cayley transform method for inverse eigenvalue problems with multiple eigenvalues, *Inverse Problems*, **31** (2015), 25 pp.
- [22] W. Shen, C. Li and X. Jin, A Ulm-like Cayley transform method for inverse eigenvalue problems with multiple eigenvalues, *Numerical Mathematics: Theory, Methods and Applications*, **9** (2016), 664-685.
- [23] W. Shen, C. Li and J. Yao, Convergence analysis of Newton-like methods for inverse eigenvalue problems with multiple eigenvalues, *SIAM Journal on Numerical Analysis*, **54** (2016), 2938-2950.
- [24] G. W. Stewart, [On the continuity of the generalized inverse](#), *SIAM Journal on Applied Mathematics*, **17** (1969), 33-45.
- [25] Y. Wang and W. Shen, An extended two-step method for inverse eigenvalue problems with multiple eigenvalues, *Numerical Mathematics: Theory, Methods and Applications*, **16** (2023), 968-992.
- [26] C. Wen, X. Chen and H. Sun, A two-step inexact Newton-Chebyshev-like method for inverse eigenvalue problems, *Linear Algebra and its Applications*, **585** (2020), 241-262.
- [27] X. Xu and J. Zhai, Inversion of trace formulas for a Sturm-Liouville operator, *Journal of Computational Mathematics*, **40** (2022), 398-416.

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