

INVERSE SOURCE PROBLEM FOR THE STOCHASTIC MAXWELL EQUATIONS IN AN INHOMOGENEOUS MEDIUM*

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Abstract. This paper concerns the inverse random source problem of the stochastic Maxwell equations driven by white noise in an inhomogeneous background medium. The well-posedness is established for the direct source problem, and the estimates and regularity of the solution are obtained. A logarithmic stability estimate is established for the inverse problem of determining the strength of the random source. The analysis only requires the random Dirichlet data at a fixed frequency.

Key words. stochastic Maxwell equations, white noise, inhomogeneous medium, inverse source problem, stability

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1. Introduction. The inverse source problem in electromagnetic waves arises in many scientific and industrial areas such as antenna design and synthesis, biomedical imaging, and optical tomography [3]. It aims to determine the unknown source from appropriate measurements of the radiating fields. For instance, the imaging modality for magnetoencephalography in medical imaging is a noninvasive neurophysiological technique which measures the electric or magnetic fields generated by neuronal activity of the brain [2, 16]. The spatial distributions of the measured fields are analyzed to localize the source currents in the brain. Driven by the important applications, inverse source problems have attracted much attention from many researchers in both the engineering and mathematical communities [10].

In most existing works, the sources have been considered to be deterministic functions for the inverse electromagnetic source scattering problems. However, in situations involving unpredictable environments, incomplete knowledge of the system, and random measurement noise, it has been realized that random sources are able to better capture the uncertain behavior of the surrounding environments [7]. Recently, uniqueness of an inverse random source problem was obtained in [13] for Maxwell equations in a homogeneous medium.

In many practical applications, the background medium in which the electromagnetic waves propagate is not homogeneous. For instance, in the medical imaging of the brain, it is important to incorporate the sudden change of sound speed across the

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human head [14]. Moreover, it is possible to achieve some desirable radiation pattern that would otherwise not be realistically possible for a source embedded in free space [15]. This phenomenon finds important applications in the antenna community in designing antenna embedding materials or substrates such as nonmagnetic dielectrics, magneto-dielectrics, and double negative meta-materials, in order to achieve specified electromagnetic radiation patterns. However, uniqueness and stability issues remain open for the inverse random source problems on Maxwell equations in inhomogeneous media.

We consider the mathematical study of the electromagnetic inverse random source problem in an inhomogeneous medium. Consider the time-harmonic Maxwell equations

$$(1.1) \quad \nabla \times \mathbf{E}(x) - ik\mathbf{H}(x) = 0, \quad \nabla \times \mathbf{H}(x) + ikn(x)\mathbf{E}(x) = \mathbf{J}(x), \quad x \in \mathbb{R}^3,$$

where $k > 0$ is the wavenumber, the positive smooth function n is the refractive index, and $m := 1 - n \in C_0^\infty(B_R)$ with $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$. $\mathbf{J}$ is the random source driven by an additive white noise of the form

$$\mathbf{J} = \dot{\mathbf{W}}_x = \sqrt{\sigma(x)}(\dot{W}_1(x), \dot{W}_2(x), \dot{W}_3(x))^\top,$$

where $\dot{W}_1(x)$, $\dot{W}_2(x)$, and $\dot{W}_3(x)$ are three independent one-dimensional white noise instances which are defined on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $\sigma \in C_0^\infty(B_R)$ is the strength of the random source. Eliminating the magnetic field $\mathbf{H}$ from (1.1), we obtain the decoupled Maxwell system for the electric field $\mathbf{E}$:

$$(1.2) \quad \nabla \times (\nabla \times \mathbf{E}) - k^2 n \mathbf{E} = ik\mathbf{J} \quad \text{in } \mathbb{R}^3.$$

In addition, the following Silver–Müller radiation condition is required at infinity to ensure the uniqueness of the direct problem:

$$(1.3) \quad \lim_{r \rightarrow \infty} ((\nabla \times \mathbf{E}) \times \hat{x} - ikr\mathbf{E}) = 0, \quad r = |x|.$$

Denote the boundary of B_R by ∂B_R . The inverse problem is to determine the strength σ of the random source from the tangential trace of the random electric field $\mathbf{E} \times \boldsymbol{\nu}$ measured on ∂B_R with $\boldsymbol{\nu}$ being the outward unit normal vector on ∂B_R .

Compared with the deterministic counterparts, if the random source is a rough field which does not exist pointwise, then the random source should be understood in the sense of distribution. As a result, the stochastic Maxwell equation should also be studied in the distributional sense. To deal with this issue, the random source was required to satisfy a divergence free condition in [13], and then the Maxwell equation (1.2) reduced to a vector-valued Helmholtz equation whose well-posedness and regularity have been studied in [12]. However, in our setting the random source is an additive white noise which does not satisfy the divergence free condition. Hence, a new method has to be developed to establish the well-posedness of the direct scattering problem. For the inverse random source problems driven by white noise, uniqueness and stability were achieved for acoustic waves in [11]. But the analysis there cannot be carried over to Maxwell equations, since the Green tensor of Maxwell equations is more singular than that of the Helmholtz equation. Due to the above difficulties, the inverse random source problem for Maxwell equations driven by an additive white noise is completely open. This work initiates the mathematical analysis of the direct and inverse source scattering problems for the stochastic Maxwell equations driven by an additive white noise.

In this paper, we establish the well-posedness for (1.2)–(1.3) in the sense of distribution and obtain the regularity for the electric field. The scattering theory is employed to investigate the resolvent of the differential operator defined on Sobolev space with negative smooth index. With the help of the resolvent estimates and the transparent boundary condition, we establish an integral equation which connects the random source and the tangential trace of the electric field on the boundary ∂B_R . Based on the integral equation, for the first time, the stability is derived for the inverse source problem of Maxwell equations driven by an additive white noise in inhomogeneous media. The proof employs Ito isometry and constructions of complex geometric optics (CGO) solutions to the associated Maxwell equation. The analysis requires the Dirichlet data only at a fixed frequency. Our methods are unified and can be applied to other stochastic wave equations in inhomogeneous media or with potential functions.

Stability for the inverse random source problem obtained in this work exhibits some features which differ significantly from its deterministic counterparts. As a matter of fact, general uniqueness does not hold for the deterministic inverse source problems of Maxwell equations even if multifrequency data is given, unless the vector-valued source is divergence free [1, 3]. This phenomenon may be attributed to the fact that for stochastic inverse source problems, it is less meaningful to characterize the source by a particular realization. It is the statistics such as the mean and variance that are of more interest in stochastic inverse problems, which are used to quantify the uncertainties of the random sources.

This paper is organized as follows. In section 2, we study the well-posedness of the direct source problem. Section 3 is devoted to the inverse random source problem. An integral equation is established which connects the random source and the electric field on the boundary. Based on the integral equation, a logarithmic stability estimate is derived for the inverse random source problem.

2. Direct scattering problem. In this section, we establish the well-posedness of the direct scattering problem (1.2)–(1.3) and give the regularity of the solution.

We introduce some function spaces. Denote the standard Sobolev spaces by $H^s(\mathbb{R}^3) := W^{s,2}(\mathbb{R}^3)$. The function spaces $H_{loc}^s := H_{loc}^s(\mathbb{R}^3)$ and $H_{comp}^s := H_{comp}^s(\mathbb{R}^3)$ are respectively defined by

$$\begin{aligned} H_{loc}^s(\mathbb{R}^3) &:= \{u : \chi u \in H^s(\mathbb{R}^3) \forall \chi \in C_0^\infty(\mathbb{R}^3)\}, \\ H_{comp}^s(\mathbb{R}^3) &:= \{u \in H^s(\mathbb{R}^3) : \exists \chi \in C_0^\infty(\mathbb{R}^3), \chi u = u\}. \end{aligned}$$

Denote $H(curl) := \{\mathbf{u} : \mathbf{u} \in L^2(\mathbb{R}^3)^3, \nabla \times \mathbf{u} \in L^2(\mathbb{R}^3)^3\}$ and then

$$H_{loc}(curl) := \{\mathbf{u} : \chi \mathbf{u} \in H(curl) \forall \chi \in C_0^\infty(\mathbb{R}^3)\}.$$

The space of all bounded linear operators between two function spaces X and Y is denoted by $\mathcal{L}(X, Y)$ with the operator norm $\|\cdot\|_{\mathcal{L}(X, Y)}$. Moreover, we use $a \lesssim b$ to stand for $a \leq Cb$, where $C > 0$ is a generic constant whose specific value is not required but should be clear from the context.

To establish the well-posedness of the direct problem, we investigate the resolvent of the Maxwell equation. We begin by studying the free resolvent in a homogeneous medium defined by

$$R_0(\lambda) = (\nabla \times (\nabla \times \cdot) - \lambda^2)^{-1}.$$

To obtain the mapping properties and estimates of the free resolvent, we consider the properties of the associated time-domain electromagnetic wave equation. Specifically, we introduce the operator $\mathcal{E}_0(t)$ which solves the equation

$$\square_0 \mathcal{E}_0(t) = 0, \quad \mathcal{E}_0(0) = 0, \quad \partial_t \mathcal{E}_0(0) = \mathbb{I},$$

where $\square_0 := \partial_t^2 + \nabla \times \nabla \times \cdot$ and $\mathbb{I}$ is the identity operator. As a consequence, given a source $\mathbf{f} \in L^2_{comp}(\mathbb{R}^3)^3$, $\mathbf{U}_0(x, t) = \mathcal{E}_0(t)(\mathbf{f})$ satisfies the following decoupled Maxwell equation:

$$\partial_t^2 \mathbf{U}_0(x, t) + \nabla \times (\nabla \times \mathbf{U}_0(x, t)) = 0$$

with

$$\mathbf{U}_0(x, 0) = 0, \quad \partial_t \mathbf{U}_0(x, 0) = \mathbf{f}.$$

It follows from [4, Theorem 3.1] that

$$(2.1) \quad \mathcal{E}_0(t)(\mathbf{f}) \in L^2((0, T), H_{loc}(\text{curl})), \quad \partial_t \mathcal{E}_0(t)(\mathbf{f}) \in L^2((0, T), L^2_{loc}(\mathbb{R}^3)^3).$$

The free resolvent $R_0(\lambda)$ can be represented by $\mathcal{E}_0(t)$ as follows:

$$R_0(\lambda) = \int_0^\infty e^{i\lambda t} \mathcal{E}_0(t) dt.$$

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be a cutoff function. The strong Huygens' principle implies that

$$(\mathcal{E}_0(t)\chi)(x) = 0 \quad \text{when } t > \sup\{|x - y| : y \in \text{supp}\chi\}.$$

Then we obtain

$$\chi R_0(\lambda)\chi = \int_0^L e^{i\lambda t} \chi \mathcal{E}_0(t) \chi dt,$$

where L is a constant with $L > \text{diam}\text{supp}\chi$. For more details on the Huygens' principle, we refer the reader to [9, section 3.1]. Furthermore, we have

$$(2.2) \quad \chi R_0(\lambda)\chi = (i\lambda)^{-1} \int_0^L \partial_t(e^{i\lambda t}) \chi \mathcal{E}_0(t) \chi dt = -(i\lambda)^{-1} \int_0^L e^{i\lambda t} \chi \partial_t \mathcal{E}_0(t) \chi dt.$$

Combining (2.1) and (2.2) we obtain that

$$\chi R_0(\lambda)\chi : L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3$$

is an analytic family of operators for $\lambda \in \mathbb{C}$ with the resolvent estimate

$$\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)} \leq \frac{C}{|\lambda|} e^{L(\text{Im}\lambda_-)},$$

where C is a positive constant and $\text{Im}\lambda_- := \max\{-\text{Im}\lambda, 0\}$.

In the following proposition, we establish the analyticity and estimates of the free resolvent $\chi R_0(\lambda)\chi$ as a family of operators from $H^s(\mathbb{R}^3)^3$ to $H^s(\mathbb{R}^3)^3$. Here $s \in \mathbb{R}$ can be any real number.

PROPOSITION 2.1. *Let $\chi \in C_0^\infty(\mathbb{R}^3)$. The free resolvent $\chi R_0(\lambda)\chi : H^s \rightarrow H^s$ is an analytic family of bounded operators for $\lambda \in \mathbb{C}$ with the resolvent estimates*

$$\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(H^s(\mathbb{R}^3)^3, H^s(\mathbb{R}^3)^3)} \leq \frac{C}{|\lambda|} e^{L(\text{Im}\lambda_-)}, \quad s \in \mathbb{R},$$

where C, L are positive constants with $L > \text{diam}\text{supp}\chi$ and $\text{Im}\lambda_- := \max\{-\text{Im}\lambda, 0\}$.

Proof. Let $\mathbf{G}(x, y)$ be the Green tensor for the Maxwell system. Explicitly, we have

$$\mathbf{G}(x, y) = i\lambda g(x, y)\mathbf{I}_3 + \frac{i}{\lambda} \nabla_x \nabla_x^\top g(x, y),$$

where $g(x, y) = \frac{1}{4\pi} \frac{e^{i\lambda|x-y|}}{|x-y|}$ is the fundamental solution of the three-dimensional Helmholtz equation and $\mathbf{I}_3$ is the 3×3 identity matrix. Therefore, the resolvent $R_0(\lambda) : L_{comp}^2(\mathbb{R}^3)^3 \rightarrow H_{loc}(curl)$ can be represented as follows:

$$(2.3) \quad R_0(\lambda)(\mathbf{f}) = \mathbf{G} * \mathbf{f} = \int_{\mathbb{R}^3} \mathbf{G}(x, y) \mathbf{f}(y) dy, \quad \mathbf{f} \in L_{comp}^2(\mathbb{R}^3)^3, \quad \lambda \in \mathbb{C}.$$

Now we intend to prove that the resolvent $\chi R_0(\lambda) \chi$ is a bounded operator from $H^s(\mathbb{R}^3)^3$ to $H^s(\mathbb{R}^3)^3$ for any $s \in \mathbb{R}$ and derive the corresponding resolvent estimates. If $\mathbf{f} \in H^1(\mathbb{R}^3)^3$ and $|\alpha| = 1$, we have

$$(2.4) \quad \begin{aligned} \partial^\alpha(\chi R_0(\lambda) \chi \mathbf{f}) &= \partial^\alpha(\chi(\mathbf{G} * (\chi \mathbf{f}))) = (\partial^\alpha \chi)(\mathbf{G} * (\chi \mathbf{f})) + \chi(\mathbf{G} * \partial^\alpha(\chi \mathbf{f})) \\ &= (\partial^\alpha \chi)(\mathbf{G} * (\chi \mathbf{f})) + \chi(\mathbf{G} * \partial^\alpha \chi \mathbf{f}) + \chi(\mathbf{G} * \partial^\alpha \mathbf{f} \chi), \end{aligned}$$

which implies $\chi R_0(\lambda) \chi \mathbf{f} \in H^1(\mathbb{R}^3)^3$. Then by induction we have $\chi R_0(\lambda) \chi \mathbf{f} \in H^n(\mathbb{R}^3)^3$ for $\mathbf{f} \in H^n(\mathbb{R}^3)^3$, where n is any positive integer. Further, using the interpolation of fractional order Sobolev spaces, i.e.,

$$H^s(\Omega) = [L^2(\Omega), H^n(\Omega)]_{s/n}, \quad n > s > 0, \quad n \in \mathbb{N},$$

we have that $\chi R_0(\lambda) \chi : H^s(\mathbb{R}^3)^3 \rightarrow H^s(\mathbb{R}^3)^3$ for all $s \geq 0$ with resolvent estimates

$$\|\chi R_0(\lambda) \chi\|_{\mathcal{L}(H^s(\mathbb{R}^3)^3, H^s(\mathbb{R}^3)^3)} \leq \frac{C}{|\lambda|} e^{L(\operatorname{Im} \lambda -)}, \quad s \geq 0.$$

Now we consider the case $s < 0$. In this case, the integral representation (2.3) should be treated as a distribution in the following sense but not a pointwise defined function:

$$\langle \mathbf{G} * \mathbf{f}, \phi \rangle = \langle \mathbf{f}, \mathbf{G} * \phi \rangle, \quad \phi \in C_0^\infty(\mathbb{R}^3)^3,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space $L^2(\mathbb{R}^3)^3$. Using the definition of convolution of distributions and the symmetry $\mathbf{G}(x, y) = \mathbf{G}(y, x)$ of the Green tensor, we obtain

$$\begin{aligned} \langle \chi R_0(\lambda) \chi \mathbf{f}, \phi \rangle &= \langle \mathbf{G} * (\chi \mathbf{f}), \chi \phi \rangle = \langle \chi \mathbf{f}, \langle \mathbf{G}(x - y), \phi \chi \rangle_x \rangle_y \\ &= \langle \chi \mathbf{f}, \langle \mathbf{G}(y - x), \phi \chi \rangle_x \rangle_y \\ &= \langle \chi \mathbf{f}, \mathbf{G} * (\phi \chi) \rangle = \langle \mathbf{f}, \chi R_0(\lambda) \chi \phi \rangle, \quad \phi \in C_0^\infty(\mathbb{R}^3)^3. \end{aligned}$$

Then letting $\varphi \in H^{-s}(\mathbb{R}^3)^3$ and approximating it by a sequence of smooth functions $\{\varphi_j\}_{j=1}^\infty$ we obtain

$$(2.5) \quad \langle \chi R_0(\lambda) \chi \mathbf{f}, \varphi_j \rangle = \langle \mathbf{f}, \chi R_0(\lambda) \chi \varphi_j \rangle.$$

As $\chi R_0(\lambda) \chi \varphi_j \in H^{-s}(\mathbb{R}^3)^3$ with $\|\chi R_0(\lambda) \chi \varphi_j\|_{H^{-s}(\mathbb{R}^3)^3} \leq \frac{C}{|\lambda|} e^{L(\operatorname{Im} \lambda -)} \|\varphi_j\|_{H^{-s}(\mathbb{R}^3)^3}$, we have

$$|\langle \mathbf{f}, \chi R_0(\lambda) \chi \varphi_j \rangle| \leq \frac{C}{|\lambda|} e^{L(\operatorname{Im} \lambda -)} \|\mathbf{f}\|_{H^s(\mathbb{R}^3)^3} \|\varphi_j\|_{H^{-s}(\mathbb{R}^3)^3}$$

which gives by letting $j \rightarrow \infty$ that the resolvent $\chi R_0(\lambda)\chi : H^s(\mathbb{R}^3)^3 \rightarrow H^s(\mathbb{R}^3)^3$ for $s < 0$ is a bounded operator with the corresponding resolvent estimate

$$\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(H^s(\mathbb{R}^3)^3, H^s(\mathbb{R}^3)^3)} \leq \frac{C}{|\lambda|} e^{L(\operatorname{Im}\lambda_-)}, \quad s < 0.$$

The proof is complete. $\square$

In the following we consider the resolvent of the Maxwell equation (1.2) in an inhomogeneous medium,

$$R(\lambda) = (c^2(x)\nabla \times (\nabla \times \cdot) - \lambda^2)^{-1}.$$

Denote $\frac{1}{n(x)} = c^2(x)$. As before, we introduce the wave operator $\mathcal{E}(t)$ which solves the wave equation

$$\square \mathcal{E}(t) = 0, \quad \mathcal{E}(0) = 0, \quad \partial_t \mathcal{E}(0) = \mathbb{I},$$

where $\square := \partial_t^2 + c^2(x)\nabla \times (\nabla \times \cdot)$. We assume that the medium satisfies the following nontrapping condition (cf. [9, Definition 4.42]).

Assumption 2.2. For any $a > R$, $\exists T_a > 0$ such that for each $\chi \in C_0^\infty(B_a)$, $\chi|_{B_{R+\tau}} = 1$ with τ being a small constant, it holds that

$$\chi \mathcal{E}(t)\chi|_{t>T_a} \in C^\infty((T_a, \infty); \mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)).$$

The following proposition concerns the analyticity and estimate of the resolvent $R(\lambda)$. The proof adapts the arguments in the derivation of [9, Theorem 4.43]. For the reader's convenience, we give the proof in the appendix.

PROPOSITION 2.3. *For any $M > 0$, there exists $C_0 > 0$ such that $R(\lambda)$ is holomorphic in the domain*

$$\Omega_M = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq -M \log |\lambda|, \quad |\lambda| \geq C_0\}.$$

Assuming $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi = 1$ near B_R , the following resolvent estimate holds:

$$(2.6) \quad \|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3} \leq \frac{C}{|\lambda|} e^{C_1(\operatorname{Im}\lambda)_-}$$

for $\lambda \in \Omega_M$, where C and C_1 are positive constants and $(\operatorname{Im}\lambda)_- := \max(0, -\operatorname{Im}\lambda)$.

Let $\mathbf{G}_n(x, y)$ be the Green tensor of the Maxwell equations in the inhomogeneous medium which satisfies

$$\nabla \times (\nabla \times \mathbf{G}_n) - k^2 n \mathbf{G}_n = \mathbf{I}_3 \delta(x - y).$$

It can be decomposed as follows:

$$\mathbf{G}_n(x, y) = \mathbf{G}(x, y) + \mathbf{P}(x, y).$$

Here $\mathbf{P}(x, y)$ is a smooth matrix with $\mathbf{P}(x, y) = \mathbf{P}(y, x)$. Using Proposition 2.3 and repeating the arguments for the proof of Proposition 2.1, we obtain the following theorem for the resolvent of the Maxwell equation (1.2) in an inhomogeneous medium.

THEOREM 2.4. *For any $M > 0$, there exists $C_0 > 0$ such that $R(\lambda)$ is holomorphic in the domain*

$$\Omega_M = \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq -M \log |\lambda|, \quad |\lambda| \geq C_0\}.$$

Assume $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi = 1$ near B_R . The following resolvent estimates hold:

$$\|\chi R(\lambda) \chi\|_{H^s(\mathbb{R}^3)^3 \rightarrow H^s(\mathbb{R}^3)^3} \leq C |\lambda|^{s-1} e^{C_1(\operatorname{Im} \lambda)_-}$$

for $s \in \mathbb{Z}$ and $\lambda \in \Omega_M$, where C and C_1 are positive constants and $(\operatorname{Im} \lambda)_- := \max\{0, -\operatorname{Im} \lambda\}$.

As a corollary of Theorem 2.4, we obtain the well-posedness of the direct problem.

THEOREM 2.5. *Assume the medium is nontrapping. The direct scattering problem (1.2)–(1.3) admits a unique solution $\mathbf{E} \in H_{loc}^{-3/2-\delta}(\mathbb{R}^3)^3$ almost surely with the estimate*

$$\|\mathbf{E}\|_{H_{loc}^{-3/2-\delta}(\mathbb{R}^3)^3} \lesssim \|\mathbf{J}\|_{H^{-3/2-\delta}(\mathbb{R}^3)^3}.$$

Proof. It is known that $\mathbf{J} \in H^{-3/2-\delta}(\mathbb{R}^3)^3$ where δ is any positive constant by the regularity of the white noise. Then the existence and estimate of the solution are a direct consequence of Theorem 2.4. The uniqueness is obtained by utilizing the uniqueness result established in [5] for the deterministic counterpart. $\square$

3. Inverse random source problem. In this section, we study the inverse random source problem. The goal is to derive a stability estimate of determining σ from the tangential trace of the electric field on ∂B_R .

Let $\mathbf{E} \times \boldsymbol{\nu}$ and $\mathbf{H} \times \boldsymbol{\nu}$ be the tangential trace of the electric field and the magnetic field, respectively. It is shown in [5] that there exists a capacity operator T_M such that

$$(3.1) \quad \mathbf{H} \times \boldsymbol{\nu} = T_M(\mathbf{E} \times \boldsymbol{\nu}) \quad \text{on } \partial B_R,$$

which implies that $\mathbf{H} \times \boldsymbol{\nu}$ can be computed once $\mathbf{E} \times \boldsymbol{\nu}$ is available on ∂B_R . The transparent boundary condition (3.1) can be equivalently written as

$$(3.2) \quad (\nabla \times \mathbf{E}) \times \boldsymbol{\nu} = ikT_M(\mathbf{E} \times \boldsymbol{\nu}) \quad \text{on } \partial B_R.$$

In what follows, we establish an integral equation connecting σ and the Dirichlet boundary data. Denote $\mathbf{f} := ik\mathbf{J}$ and $\mathbf{f}_\varepsilon := \frac{1}{\varepsilon^3} \rho(\frac{\cdot}{\varepsilon}) * \mathbf{f}$, where the mollifier ρ is a compactly supported smooth function such that $\int_{\mathbb{R}^3} \rho(x) dx = 1$. Since $\mathbf{f} \in H^{-3/2-\delta}(\mathbb{R}^3)^3$ with compact support, $\mathbf{f}_\varepsilon$ is smooth and $\mathbf{f}_\varepsilon \rightarrow \mathbf{f}$ in $H^{-3/2-\delta}(\mathbb{R}^3)^3$. For each $\mathbf{f}_\varepsilon$, there exists a unique radiating solution $\mathbf{E}_\varepsilon$ to (1.2) such that

$$(3.3) \quad \nabla \times (\nabla \times \mathbf{E}_\varepsilon) - k^2 n \mathbf{E}_\varepsilon = \mathbf{f}_\varepsilon \quad \text{in } \mathbb{R}^3.$$

Let $\mathbf{U} \in C^\infty(\overline{B_R})^3$ be a solution to the homogeneous Maxwell equation

$$\nabla \times (\nabla \times \mathbf{U}) - k^2 n \mathbf{U} = 0 \quad \text{in } B_R.$$

Multiplying both sides of the equation by $\mathbf{U}$ and integrating by parts yields

$$\int_{B_R} \mathbf{f}_\varepsilon \cdot \mathbf{U} dx = - \int_{\partial B_R} ikT_M(\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot \mathbf{U} + (\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x).$$

Then by letting $\varepsilon \rightarrow 0$ we obtain

$$(3.4) \quad \lim_{\varepsilon \rightarrow 0} \int_{B_R} \mathbf{f}_\varepsilon \cdot \mathbf{U} dx = \int_{B_R} \mathbf{f} \cdot \mathbf{U} dx.$$

We also have

$$(3.5) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\partial B_R} ikT_M(\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot \mathbf{U} + (\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x) \\ = \int_{\partial B_R} ikT_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U} + (\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x). \end{aligned}$$

To prove the above limit, note that

$$\begin{aligned} & \left| \int_{\partial B_R} ikT_M(\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot \mathbf{U} - ikT_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U} ds(x) \right| \\ & \leq \|ikT_M(\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) - ikT_M(\mathbf{E} \times \boldsymbol{\nu})\|_{L^2(\partial B_R)^3} \|\mathbf{U}\|_{L^2(\partial B_R)^3} \\ & \leq \|\mathbf{E}_\varepsilon - \mathbf{E}\|_{H^{3/2}(\Omega)^3} \|\mathbf{U}\|_{H^{3/2}(\partial B_R)^3}. \end{aligned}$$

Here Ω is a bounded domain satisfying $\partial B_R \subset \Omega$ and $\Omega \cap \text{supp} \mathbf{J} = \emptyset$. The first inequality is derived by using the Cauchy–Schwarz inequality, and to obtain the second inequality we apply (3.2) and the standard trace theorem in Sobolev spaces. Notice that near ∂B_R the Maxwell equation reduces to the Helmholtz equation, which implies

$$-\Delta(\mathbf{E}_\varepsilon - \mathbf{E}) - k^2(\mathbf{E}_\varepsilon - \mathbf{E}) = 0.$$

Then from the resolvent estimate in Theorem 2.4 by letting $s = -3/2 - \delta$ and by the interior elliptic regularity theory [17, (7.13)], we have for any $N \in \mathbb{N}$

$$\begin{aligned} & \|\mathbf{E}_\varepsilon - \mathbf{E}\|_{H^{-3/2-\delta+2N}(\Omega)^3} \\ & \leq C(N) \|\mathbf{E}_\varepsilon - \mathbf{E}\|_{H^{-3/2-\delta}(\Omega)^3} \\ & \leq C(N) \|\mathbf{f}_\varepsilon - \mathbf{f}\|_{H^{-3/2-\delta}(B_R)^3} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Thus, we obtain the following limit by letting $N = 2$:

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_R} T_M(\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot \mathbf{U} ds(x) = \int_{\partial B_R} T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U} ds(x).$$

In a similar way we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_R} (\mathbf{E}_\varepsilon \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x) = \int_{\partial B_R} (\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x),$$

which yields (3.5). Combining (3.4) and (3.5) we establish the following integral equation:

$$(3.6) \quad \int_{B_R} \mathbf{f} \cdot \mathbf{U} dx = \int_{\partial B_R} ikT_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U} + (\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}) ds(x).$$

Based on the integral identity (3.6), we obtain the following lemma, which plays a crucial role in the stability estimate. It establishes an estimate of the strength of the random source by the measurement of the electric field.

LEMMA 3.1. For two functions $\mathbf{U}_1$ and $\mathbf{U}_2$ which are smooth in $\overline{B_{R'}}$ and satisfy the Maxwell equations

$$(3.7) \quad \nabla \times (\nabla \times \mathbf{U}_j) - k^2 n \mathbf{U}_j = 0 \quad \text{in } B_{R'}, \quad j = 1, 2,$$

there exists a positive constant C dependent on k and R such that

$$(3.8) \quad \left| \int_{\mathbb{R}^3} \sigma \mathbf{U}_1^\top \mathbf{U}_2 dx \right| \leq C \epsilon \|\mathbf{U}_1\|_{L^2(B_{R'})^3} \|\mathbf{U}_2\|_{L^2(B_{R'})^3},$$

where $R' > R$ and the boundary data ϵ is given by

$$\epsilon = \max\{\|\mathbb{F}_1\|_{\partial B_R \times \partial B_R}, \|\mathbb{F}_2\|_{\partial B_R \times \partial B_R}, \|\mathbb{F}_3\|_{\partial B_R \times \partial B_R}\}$$

with the norm $\|\cdot\|_{\partial B_R \times \partial B_R}$ for matrix-valued functions defined as

$$\|\mathbb{A}\|_{\partial B_R \times \partial B_R} := \sum_{i,j=1}^3 \|A_{ij}\|_{L^2(\partial B_R \times \partial B_R)}.$$

Here

$$\begin{aligned} \mathbb{F}_1(x, y) &:= \mathbb{E}[(\mathbf{E}(x) \times \boldsymbol{\nu}(x))(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top], \\ \mathbb{F}_2(x, y) &:= \mathbb{E}[T_M(\mathbf{E}(x) \times \boldsymbol{\nu}(x))(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top], \\ \mathbb{F}_3(x, y) &:= \mathbb{E}[T_M(\mathbf{E}(x) \times \boldsymbol{\nu}(x))T_M(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top]. \end{aligned}$$

Proof. Using the Ito isometry we obtain

$$(3.9) \quad \mathbb{E} \left[\int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{U}_1 dx \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{U}_2 dx \right] = \int_{\mathbb{R}^3} \sigma \mathbf{U}_1^\top \mathbf{U}_2 dx.$$

On the other hand, by (3.6) we have

$$(3.10) \quad ik \int_{B_R} \mathbf{J} \cdot \mathbf{U}_j dx = \int_{\partial B_R} ik T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U}_j + (\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{U}_j) ds(x), \quad j = 1, 2.$$

Then combining (3.9) and (3.10) we obtain

$$\begin{aligned} & -k^2 \int_{\mathbb{R}^3} \sigma \mathbf{U}_1^\top \mathbf{U}_2 dx \\ &= \mathbb{E} \left[\int_{\partial B_R} (\nabla \times \mathbf{U}_1) \cdot (\mathbf{E} \times \boldsymbol{\nu}) + ik T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U}_1 ds(x) \right. \\ & \quad \times \left. \int_{\partial B_R} (\nabla \times \mathbf{U}_2) \cdot (\mathbf{E} \times \boldsymbol{\nu}) + ik T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \mathbf{U}_2 ds(y) \right] \\ &= \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[(\mathbf{E}(x) \times \boldsymbol{\nu}(x))(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top] \\ & \quad \otimes ((\nabla \times \mathbf{U}_1(x))(\nabla \times \mathbf{U}_2(y))^\top) ds(x) ds(y) \\ & \quad + ik \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[T_M(\mathbf{E}(x) \times \boldsymbol{\nu}(x))(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top] \\ & \quad \otimes (\mathbf{U}_1(x)(\nabla \times \mathbf{U}_2(y))^\top) ds(x) ds(y) \end{aligned}$$

$$\begin{aligned}
 & + ik \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[(\mathbf{E}(x) \times \boldsymbol{\nu}(x)) T_M(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top] \\
 & \quad \otimes ((\nabla \times \mathbf{U}_1(x))(\mathbf{U}_2(y))^\top) ds(x) ds(y) \\
 & - k^2 \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[T_M(\mathbf{E}(x) \times \boldsymbol{\nu}(x)) T_M(\mathbf{E}(y) \times \boldsymbol{\nu}(y))^\top] \\
 & \quad \otimes (\mathbf{U}_1(x) \mathbf{U}_2(y)^\top) ds(x) ds(y).
 \end{aligned}$$

Then we derive the following estimate by an application of the Cauchy–Schwarz inequality:

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^3} \sigma \mathbf{U}_1^\top \mathbf{U}_2 dx \right| \\
 & \leq C \epsilon \left(\|\mathbf{U}_1\|_{L^2(\partial B_R)^3} \|\mathbf{U}_2\|_{L^2(\partial B_R)^3} + \|\nabla \times \mathbf{U}_1\|_{L^2(\partial B_R)^3} \|\mathbf{U}_2\|_{L^2(\partial B_R)^3} \right. \\
 & \quad \left. + \|\mathbf{U}_1\|_{L^2(\partial B_R)^3} \|\nabla \times \mathbf{U}_2\|_{L^2(\partial B_R)^3} + \|\nabla \times \mathbf{U}_1\|_{L^2(\partial B_R)^3} \|\nabla \times \mathbf{U}_2\|_{L^2(\partial B_R)^3} \right),
 \end{aligned}$$

where C is dependent on k and R . Then since equations (3.7) reduce to Helmholtz equations $(\Delta + k^2)\mathbf{U}_j = 0$ near ∂B_R , by applying the trace theorem and standard interior regularity estimate of elliptic equations we obtain (3.8). The proof is complete. $\square$

Based on the estimate (3.8), in the following we derive a stability estimate of recovering the strength σ from the correlation data. To this end, we shall choose appropriate test functions $\mathbf{U}_1$ and $\mathbf{U}_2$ which are solutions to the associated Maxwell equation (3.7).

The following lemma given by [8, Theorem 4] provides us with useful CGO solutions.

LEMMA 3.2. *Let $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{C}^3$ such that $\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = k^2$, $\boldsymbol{\zeta} \cdot \boldsymbol{\eta} = 0$, and $|\Im \boldsymbol{\zeta}| \geq M_1$. Then there exists a solution to*

$$(3.11) \quad \nabla \times (\nabla \times \mathbf{U}) - k^2 n \mathbf{U} = 0 \quad \text{in } B_{R'}$$

satisfying $\mathbf{U}(x, \boldsymbol{\zeta}, \boldsymbol{\eta}) = e^{i\boldsymbol{\zeta} \cdot x} (\boldsymbol{\eta} + f(x, \boldsymbol{\zeta}, \boldsymbol{\eta}) \boldsymbol{\zeta} + \mathbf{V}(x, \boldsymbol{\zeta}, \boldsymbol{\eta})) \in C^\infty(\overline{B_{R'}})^3$, where the function and the vector field satisfy

$$(3.12) \quad \|f(\cdot, \boldsymbol{\zeta}, \boldsymbol{\eta})\|_{L^2(B_{R'})^3} + \|\mathbf{V}(\cdot, \boldsymbol{\zeta}, \boldsymbol{\eta})\|_{L^2(B_{R'})^3} \leq \frac{M_2 |\boldsymbol{\eta}|}{|\Im \boldsymbol{\zeta}|}.$$

Here M_1 and M_2 are two constants depending on n , k , and R' with $R' > R$.

It should be noticed that, in order to get the smoothness of $\mathbf{U}$ in $B_{R'}$, we use the standard interior elliptic regularity and the fact that (3.11) implies

$$\Delta \mathbf{U} + \nabla \left(\frac{\nabla n \cdot \mathbf{U}}{n} \right) + k^2 n \mathbf{U} = 0.$$

Our main result on stability of the inverse random source problem is stated as follows.

THEOREM 3.3. *Assume σ satisfies the a priori regularity conditions*

$$\|\sigma\|_{L^\infty(\mathbb{R}^3)}, \|\sigma\|_{H^s(\mathbb{R}^3)} \leq Q$$

with $s > 0$ and $Q > 0$ being two positive constants. Then there holds the stability estimate

$$\|\sigma\|_{L^2(\mathbb{R}^3)}^2 \leq C \frac{1}{(-\log \epsilon)^{\frac{2s}{7+2s}}},$$

where ϵ is defined in Lemma 3.1 and C is a positive constant dependent on n, k, Q, M_1, M_2 . Here M_1, M_2 are constants specified in Lemma 3.2

Proof. We choose ζ and η as follows. For any given $\xi \in \mathbb{R}^3$, take $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^2$ such that $\{\hat{\xi}, \mathbf{d}_1, \mathbf{d}_2\}$ forms an orthonormal basis of $\mathbb{R}^3$. Under this basis, set

$$\begin{aligned} \zeta_1 &= \left(-\frac{|\xi|}{2}, i\sqrt{t^2 - k^2 + \frac{|\xi|^2}{4}}, t \right), & \zeta_2 &= \left(-\frac{|\xi|}{2}, -i\sqrt{t^2 - k^2 + \frac{|\xi|^2}{4}}, -t \right), \\ \eta_1 &= \left(1, 0, \frac{|\xi|}{2t} \right), & \eta_2 &= \left(1, 0, -\frac{|\xi|}{2t} \right), \end{aligned}$$

where t is a parameter satisfying $t \geq M_1 + 2k$. Then it follows from Lemma 3.2 that there exist CGO solutions $\mathbf{U}_j = e^{i\zeta_j \cdot x}(\eta_j + f_j(x, \zeta_j, \eta_j)\zeta_j + \mathbf{V}_j(x, \zeta_j, \eta_j))$, $j = 1, 2$, satisfying

$$(3.13) \quad \|f(\cdot, \zeta_j, \eta_j)\|_{L^2(B_{R'})} + \|\mathbf{V}(\cdot, \zeta_j, \eta_j)\|_{L^2(B_{R'})^3} \leq \frac{M_2|\eta_j|}{|\Im \zeta_j|} \leq \frac{2M_2 + \frac{M_2|\xi|}{M_1}}{t} \leq C \frac{(1 + |\xi|)}{t}.$$

A direct calculation gives

$$\begin{aligned} \|\mathbf{U}_j\|_{L^2(B_{R'})^3} &\leq \|e^{i\zeta_j \cdot x} \eta_j\|_{L^2(B_{R'})^3} + \|e^{i\zeta_j \cdot x} f \zeta_j\|_{L^2(B_{R'})^3} + \|e^{i\zeta_j \cdot x} \mathbf{V}\|_{L^2(B_{R'})^3} \\ &\leq C e^{(t+|\xi|)R'} \left(1 + \frac{|\xi|}{t} \right). \end{aligned}$$

Inserting the CGO solutions constructed above into (3.8) yields

$$(3.14) \quad \left| \int_{\mathbb{R}^3} \sigma \mathbf{U}_1 \cdot \mathbf{U}_2 \, dx \right| \leq C e^{2(t+|\xi|)R'} \left(1 + \frac{|\xi|}{t} \right)^2 \epsilon.$$

Notice that

$$\mathbf{U}_1(x, \zeta, \eta) \cdot \mathbf{U}_2(x, \zeta, \eta) = e^{-i\xi \cdot x} \left(1 - \frac{|\xi|^2}{4t^2} + r(x) \right),$$

where $r(x)$ is given by

$$\begin{aligned} r(x) &= -|\xi|(f(x, \zeta_1, \eta_1) + f(x, \zeta_2, \eta_2)) + f(x, \zeta_1, \eta_1)f(x, \zeta_2, \eta_2)(|\xi|^2/2 - k^2) \\ &\quad + \mathbf{V}(x, \zeta_1, \eta_1) \cdot \mathbf{V}(x, \zeta_2, \eta_2) + \eta_1 \cdot \mathbf{V}(x, \zeta_2, \eta_2) + \eta_2 \cdot \mathbf{V}(x, \zeta_1, \eta_1) \\ &\quad + f(x, \zeta_2, \eta_2)\eta_2 \cdot \mathbf{V}(x, \zeta_1, \eta_1) + f(x, \zeta_1, \eta_1)\eta_1 \cdot \mathbf{V}(x, \zeta_2, \eta_2). \end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|r\|_{L^1(B_{R'})} &\leq |\xi|(\|f(x, \zeta_1, \eta_1)\|_{L^1(B_{R'})} + \|f(x, \zeta_2, \eta_2)\|_{L^1(B_{R'})}) \\ &\quad + \|f(x, \zeta_1, \eta_1)\|_{L^2(B_{R'})}\|f(x, \zeta_2, \eta_2)\|_{L^2(B_{R'})}||\xi|^2/2 - k^2| \\ &\quad + \|\mathbf{V}(x, \zeta_1, \eta_1)\|_{L^2(B_{R'})^3}\|\mathbf{V}(x, \zeta_2, \eta_2)\|_{L^2(B_{R'})^3} \end{aligned}$$

$$\begin{aligned}
 & + |\boldsymbol{\eta}_1| \| \mathbf{V}(x, \boldsymbol{\zeta}_2, \boldsymbol{\eta}_2) \|_{L^1(B_{R'})^3} + |\boldsymbol{\eta}_2| \| \mathbf{V}(x, \boldsymbol{\zeta}_1, \boldsymbol{\eta}_1) \|_{L^1(B_{R'})^3} \\
 & + |\boldsymbol{\eta}_2| \| f(x, \boldsymbol{\zeta}_2, \boldsymbol{\eta}_2) \|_{L^2(B_{R'})} \| \mathbf{V}(x, \boldsymbol{\zeta}_1, \boldsymbol{\eta}_1) \|_{L^2(B_{R'})^3} \\
 & + |\boldsymbol{\eta}_1| \| f(x, \boldsymbol{\zeta}_1, \boldsymbol{\eta}_1) \|_{L^2(B_{R'})} \| \mathbf{V}(x, \boldsymbol{\zeta}_2, \boldsymbol{\eta}_2) \|_{L^2(B_{R'})^3}.
 \end{aligned}$$

As $t > M_1 + 2k$, it is clear that

$$|\boldsymbol{\eta}_j| = \sqrt{1 + \frac{|\xi|^2}{4t^2}} \leq C \left(1 + \frac{|\xi|}{t} \right) \leq C(1 + |\xi|), \quad j = 1, 2.$$

Using (3.13) and noting $t \geq M_1 + 2k$, we obtain

$$\begin{aligned}
 \|r\|_{L^1(B_{R'})} & \leq C \left(\left(1 + k^2 + \frac{|\xi|^2}{2} + |\xi| \right) \left(\frac{1 + |\xi|}{t} \right)^2 + (1 + |\xi|) \frac{(1 + |\xi|)}{t} \right) \\
 (3.15) \quad & \leq C \left((1 + |\xi|^2) \frac{(1 + |\xi|^2)}{t^2} + (1 + |\xi|) \frac{(1 + |\xi|)}{t} \right) \\
 & \leq C \left(\frac{1 + |\xi|^4}{t^2} + \frac{1 + |\xi|^2}{t} \right) \leq C \frac{(1 + |\xi|^4)}{t}.
 \end{aligned}$$

Since $\|\sigma\|_{L^\infty(\mathbb{R}^3)} \leq Q$, combining (3.14) and (3.15) yields

$$|\widehat{\sigma}(\xi)| = \left| \int_{\mathbb{R}^3} \sigma(x) e^{-i\xi \cdot x} dx \right| \leq C \left(e^{2(t+|\xi|)R'} \left(1 + \frac{|\xi|}{t} \right)^2 \epsilon + \frac{1 + |\xi|^4}{t} \right).$$

Then noting $\|\sigma\|_{H^s(\mathbb{R}^3)} \leq Q$ we arrive at

$$\begin{aligned}
 \|\sigma\|_{L^2(\mathbb{R}^3)}^2 & \leq C \left(\left| \int_{|\xi| < \rho} |\widehat{\sigma}(\xi)|^2 d\xi \right| + \left| \int_{|\xi| \geq \rho} |\widehat{\sigma}(\xi)|^2 d\xi \right| \right) \\
 & \leq C \left(\rho^3 \left(e^{2(t+\rho)R'} \left(1 + \frac{\rho}{t} \right)^2 \epsilon + \frac{1 + \rho^4}{t} \right) + \frac{\|\sigma\|_{H^s(\mathbb{R}^3)}^2}{\rho^{2s}} \right) \\
 & \leq C \left(\rho^3 e^{2(t+\rho)R'} \left(1 + \frac{\rho}{t} \right)^2 \epsilon + \frac{\rho^7}{t} + \frac{1}{\rho^{2s}} \right).
 \end{aligned}$$

Here $\rho > 1$ is a positive constant. By letting $\rho = t^{\frac{1}{7+2s}}$ we obtain for $t > 1$

$$\begin{aligned}
 \|\sigma\|_{L^2(\mathbb{R}^3)}^2 & \leq C \left(t^{\frac{3}{7+2s}} e^{2(t+t^{\frac{1}{7+2s}})R'} \left(1 + t^{-\frac{6+2s}{7+2s}} \right)^2 \epsilon + t^{-\frac{2s}{7+2s}} \right) \\
 & \leq C \left(t^{\frac{3}{7+2s}} e^{4tR'} \epsilon + t^{-\frac{2s}{7+2s}} \right).
 \end{aligned}$$

Without loss of generality, by assuming ϵ is a sufficiently small constant, we are able to choose $t = -\frac{1}{2R'} \log \epsilon$, which yields $t > \{1, M_1 + 2k\}$ and $\rho = t^{\frac{1}{7+2s}} > 1$. As a result, we derive the stability estimate

$$\|\sigma\|_{L^2(\mathbb{R}^3)}^2 \leq C \frac{1}{(-\log \epsilon)^{\frac{2s}{7+2s}}},$$

and it can be seen from the proof that the constant C depends on n, k, Q, M_1, M_2 . The proof is complete. $\square$

Remark 3.4. The stability implies the uniqueness of the inverse random source problems for the Maxwell equation at a fixed frequency. It is of logarithmic type which illustrates that the inverse problem is ill-posed. Moreover, the stability increases if the regularity of the strength improves.

4. Conclusion. In this paper, we have investigated the inverse random source problem for Maxwell equations in an inhomogeneous medium. We have established the well-posedness of the direct problem and obtained the estimates and regularity of the solution. By using the CGO solutions to the associated Maxwell equation, we have obtained a logarithmic stability of determining the strength of the random source by using Dirichlet data only at a single frequency. The methods developed in this work can be used to study other stochastic wave equations in inhomogeneous media or with potential functions such as the elastic wave equation and Schrödinger equations. A more challenging direction is to study the case where the strength functions are different for each component of the random source. In this case, the constructions of the CGO solutions are much more complicated. We hope to be able to report progress on these problems in future works.

Appendix A. Proof of Theorem 2.3. In this section, we present a proof for Theorem 2.3 which follows the arguments in the derivation of [9, Theorem 4.43].

Introduce the wave operator $\mathcal{E}(t)$ which solves the Maxwell equations in an inhomogeneous medium,

$$\square \mathcal{E}(t) = 0, \quad \mathcal{E}(0) = 0, \quad \partial_t \mathcal{E}(0) = \mathbb{I},$$

where $\square := \partial_t^2 + P$ with $P = c^2(x) \nabla \times \nabla \times \cdot$ and $\mathbb{I}$ is the identity operator. Thus, given a source $\mathbf{f}$, $\mathbf{U}(x, t) = \mathcal{E}(t)(\mathbf{f})$ satisfies the following decoupled Maxwell equation:

$$(\partial_t^2 + c^2(x) \nabla \times \nabla \times) \mathbf{U}(x, t) = 0$$

with

$$\mathbf{U}(x, 0) = 0, \quad \partial_t \mathbf{U}(x, 0) = \mathbf{f}.$$

It holds that

$$(A.1) \quad \mathcal{E}(t)(\mathbf{f}) \in L^2((0, T), H_{loc}(\text{curl})), \quad \partial_t \mathcal{E}(t)(\mathbf{f}) \in L^2((0, T), L_{loc}^2(\mathbb{R}^3)^3).$$

Recall $\square_0 = \partial_t^2 + \nabla \times \nabla \times \cdot$. We use the notation $[A, B] = AB - BA$ for commutator.

Choose a large enough such that $\text{supp } \chi \subset B_a$, and take $\chi_a \in C_c^\infty(B_a)$, $\chi_a \equiv 1$ on $\text{supp } \chi$. Assume that $\psi_a \in C^\infty(\mathbb{R}_t \times \mathbb{R}^3)$ satisfies

$$\text{supp } \psi_a \cap (\mathbb{R}_t \times \mathbb{R}^3 \setminus B_R) \subset \{(t, x) : |x| + T_a \leq t \leq |x| + T_a + 1\},$$

and

$$\psi_a|_{\mathbb{R}_t \times B_{R+\epsilon}} = \psi_a^0(t),$$

where $\psi_a^0 \in C_c^\infty(T_a + R, T_a + R + 1)$ is a function of t only. First we show the following result.

LEMMA A.1. $\mathcal{E}(t)$ satisfies

$$(A.2) \quad \psi_a \mathcal{E}(t) \chi_a \in C^\infty(\mathbb{R}_t; \mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)).$$

Proof. Let $\chi_0, \chi_1 \in C_0^\infty(B_a)$ such that $\chi_0 \equiv 1$ on $\text{supp } \chi_a$ and $\chi_1 \equiv 1$ on $\text{supp } \chi_0$. The nontrapping condition implies

$$(A.3) \quad \chi_0 \mathcal{E}(t) \chi_a|_{t > T_a} \in C^\infty((T_a, \infty); \mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)).$$

Notice $\chi_0 \chi_1 = \chi_0$. Using the relations

$$(c^2(x) - 1)[\nabla \times \nabla \times \cdot, \chi_0] = 0, \quad \chi_0[\nabla \times \nabla \times \cdot, \chi_1] = 0,$$

we obtain

$$\begin{aligned}
 & (\partial_t^2 - c^2 \nabla \times \nabla \times)((1 - \chi_0)\mathcal{E}(t)\chi_a) \\
 &= -\chi_0 \partial_t^2(\mathcal{E}(t)\chi_a) + c^2 \nabla \times \nabla \times (\chi_0 \mathcal{E}(t)\chi_a) \\
 &= -\chi_0 \partial_t^2(\mathcal{E}(t)\chi_a) + \nabla \times \nabla \times (\chi_0 \mathcal{E}(t)\chi_a) + c^2 \chi_0 \nabla \times \nabla \times (\mathcal{E}(t)\chi_a) \\
 &= -\chi_0 \nabla \times \nabla \times (\mathcal{E}(t)\chi_a) \\
 &= \nabla \times \nabla \times (\chi_0 \mathcal{E}(t)\chi_a) - \chi_0 \nabla \times \nabla \times (\mathcal{E}(t)\chi_a) \\
 &= \nabla \times \nabla \times (\chi_0 \chi_1 \mathcal{E}(t)\chi_a) - \chi_0 \chi_1 \nabla \times \nabla \times (\mathcal{E}(t)\chi_a) \\
 &= [\nabla \times \nabla \times, \chi_0](\chi_1 \mathcal{E}(t)\chi_a) + \chi_0 \nabla \times \nabla \times (\chi_1 \mathcal{E}(t)\chi_1) - \chi_0 \chi_1 \nabla \times \nabla \times (\mathcal{E}(t)\chi_a) \\
 &= [\nabla \times \nabla \times, \chi_0](\chi_1 \mathcal{E}(t)\chi_a) + \chi_0 \chi_0 [\nabla \times \nabla \times, \chi_1](\nabla \times \nabla \times (\mathcal{E}(t)\chi_a)) \\
 &= [\nabla \times \nabla \times, \chi_0](\chi_1 \mathcal{E}(t)\chi_a) \\
 &= -[\partial_t^2 - \nabla \times \nabla \times, \chi_0](\chi_1 \mathcal{E}(t)\chi_a) \\
 &= -[\square_0, \chi_0](\chi_1 \mathcal{E}(t)\chi_a) \\
 &=: G_a(t).
 \end{aligned}$$

It is clear that $(1 - \chi_0)\mathcal{E}(t)\chi_a$ satisfies the initial conditions

$$(1 - \chi_0)\mathcal{E}(0)\chi_a = 0, \quad (1 - \chi_0)\partial_t \mathcal{E}(0)\chi_a = 0$$

and

$$G_a(t)|_{t>T_a} \in C^\infty((T_a, \infty); \mathcal{L}(L^2(\mathbb{R}^3)^3, H_{comp}^1(B_a \setminus \overline{B_R})^3)).$$

By the Duhamel principle and the support property of the propagator U_0 of free-space wave equations in odd dimensions

$$\text{supp } \mathcal{E}_0(t, \cdot) = \{(x, y); |x - y| = |t|\},$$

we have

(A.4)

$$\psi_a(1 - \chi_0)\mathcal{E}(t)\chi_a = \psi_a \int_{T_a}^t \mathcal{E}_0(t - s)G_a(s)ds \in C^\infty((T_a, \infty); \mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)).$$

Since $\psi \equiv 0$ on $\{t < T_a + R\}$, (A.3) and (A.4) imply (A.2). The proof is completed. $\square$

Take a function $\zeta_a \in C^\infty$ such that

$$\begin{aligned}
 \zeta_a(x, t)|_{|x|>R} &= \begin{cases} 1, & t \leq |x| + T_a, \\ 0, & t \geq |x| + T_a + 1, \end{cases} \\
 \zeta_a(x, t)|_{|x|<R+\epsilon} &= \zeta_a^0(t) = \begin{cases} 1, & t \leq R + T_a + \epsilon, \\ 0, & t \geq R + T_a + 1. \end{cases}
 \end{aligned}$$

It follows from the above claim that

$$F_a(t) = [\square, \zeta_a](\mathcal{E}(t)\chi_a) \in C^\infty(\mathbb{R}; \mathcal{L}(L^2(\mathbb{R}^3)^3, L^2(\mathbb{R}^3)^3)).$$

Notice that $\zeta_a(x, t) \equiv 1$ for $t \leq 0$, thus $[\square, \zeta_a] = 0$ on $\{t \leq 0\}$ and consequently $F_a(t) \equiv 0$ for $t \leq 0$.

Choose $\chi_b \in C_c^\infty(B_a)$ such that $\chi_b = 1$ near B_R and $\chi_a \equiv 1$ on $\text{supp } \chi_b$. Let $W_a(t)$ solve

$$\square_0 W_a(t) = -(1 - \chi_b)F_a(t), \quad W_a(t) \equiv 0, \quad t \leq 0.$$

It is clear that $W_a(t)g \in C^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3)^3)$.

LEMMA A.2. For $g \in L^2(\mathbb{R}^3)^3$

$$(A.5) \quad \text{supp}(W_a(t)g)(x) \subset \{(t, x) : |x| + T_a \leq t \leq |x| + C_a\}.$$

Here $C_a = 2a + T_a + 1$.

Proof. Write

$$-W_a(t) = (1 - \chi_b)\zeta_a H(t)\mathcal{E}(t)\chi_a - H(t)U_0(t)(1 - \chi_b)\chi_a + Q_a(t),$$

where $H(t)$ is the Heaviside function. It is easy to see that the first term has a support in $\{(t, x) : 0 \leq t \leq |x| + C_a\}$. The last term $Q_a(t)$ solves

$$\begin{aligned} \square_0 Q_a(t) &= -\square_0 W_a(t) - \square_0((1 - \chi_b)\zeta_a H(t)\mathcal{E}(t)\chi_a) + \square_0(H(t)U_0(t)(1 - \chi_b)\chi_a) \\ &= -\square_0 W_a(t) - \square_0((1 - \chi_b)\zeta_a H(t)\mathcal{E}(t)\chi_a) + \square_0(H(t)U_0(t)(1 - \chi_b)\chi_a) \\ &= -\square_0 W_a(t) - (1 - \chi_b)\square(\zeta_a H(t)\mathcal{E}(t)\chi_a) + [\square_0, \chi_b]\zeta_a H(t)\mathcal{E}(t)\chi_a \\ &\quad + \square_0(H(t)U_0(t)(1 - \chi_b)\chi_a) \\ &= -\square_0 W_a(t) - (1 - \chi_b)[\square, \zeta_a]H(t)\mathcal{E}(t)\chi_a - \zeta_a(1 - \chi_b)\square(H(t)\mathcal{E}(t)\chi_a) \\ &\quad + \square + [\square_0, \chi_b]\zeta_a H(t)\mathcal{E}(t)\chi_a + \square_0(H(t)U_0(t)(1 - \chi_b)\chi_a) \\ &= (1 - \chi_b)F_a(t) - (1 - \chi_b)F_a(t) - (1 - \chi_b)\delta(t)\chi_a \\ &\quad + \square + [\square_0, \chi_b]\zeta_a H(t)\mathcal{E}(t)\chi_a + \delta(t)(1 - \chi_b)\chi_a \\ &= -[\nabla \times \nabla \times, \chi_b]\zeta_a H(t)\mathcal{E}(t)\chi_a. \end{aligned}$$

Observe that

$$[\nabla \times \nabla \times, \chi_b]\zeta_a H(t)\mathcal{E}(t)\chi_a \in L^2_{\text{comp}}(B_a \setminus B_R)$$

vanishes on $\{t \geq a + T_a + 1\}$. Therefore it follows from the sharp Huygens principle that

$$\text{supp}(Q_a(t)g)(x) \subset \{(t, x) : 0 \leq t \leq |x| + C_a\}.$$

Notice also that $(1 - \chi_b)F_a(t)g = 0$ on $\{t \leq |x| + T_a\}$, by finite speed of propagation, we have

$$\text{supp}(W_a(t)g)(x) \subset \{(t, x) : |x| + T_a \leq t\},$$

which completes the lemma. $\square$

Choose $\chi_c \in C_c^\infty(B_a)$ such that $\chi_c = 1$ near B_R and $\chi_b = 1$ on $\text{supp}\chi_c$. Let

$$R_a^\#(\lambda) = \mathcal{F}_{t \rightarrow \lambda}(\zeta_a H(t)\mathcal{E}(t)\chi_a + (1 - \chi_c)W_a(t)).$$

A straightforward calculation yields

$$\begin{aligned} (P - \lambda^2)R_a^\#(\lambda) &= \mathcal{F}_{t \rightarrow \lambda}(\square(\zeta_a H(t)\mathcal{E}(t)\chi_a) + \square((1 - \chi_c)W_a(t))) \\ &= \mathcal{F}_{t \rightarrow \lambda}([\square, \zeta_a](H(t)\mathcal{E}(t)\chi_a) + \zeta_a \square(H(t)\mathcal{E}(t)\chi_a) + \square_0((1 - \chi_c)W_a(t))) \\ &= \mathcal{F}_{t \rightarrow \lambda}(F_a(t) + \delta(t)\chi_a + [\square_0, 1 - \chi_c]W_a(t) + (1 - \chi_c)\square_0 W_a(t)) \\ &= \mathcal{F}_{t \rightarrow \lambda}(F_a(t) + \delta(t)\chi_a + [\nabla \times \nabla \times, \chi_c]W_a(t) - (1 - \chi_c)(1 - \chi_b)F_a(t)) \\ &= \mathcal{F}_{t \rightarrow \lambda}(\chi_b F_a(t) + \delta(t)\chi_a + [\nabla \times \nabla \times, \chi_c]W_a(t)) \\ &= \chi_a(I + K_a(\lambda)), \end{aligned}$$

where

$$K_a(\lambda) = \mathcal{F}_{t \rightarrow \lambda}(\chi_b F_a(t) + [\nabla \times \nabla \times, \chi_c] W_a(t)).$$

We prove (2.6) by establishing the following estimates:

$$(A.6) \quad \chi R_a^\#(\lambda) = \mathcal{O}(e^{C(\operatorname{Im} \lambda) - \langle \lambda \rangle^{-1}})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3},$$

$$(A.7) \quad K_a(\lambda) = \mathcal{O}(e^{C(\operatorname{Im} \lambda) - \langle \lambda \rangle^{-N}})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3}$$

for any N . Then we have

$$(A.8) \quad R(\lambda) \chi_a = R_a^\#(\lambda)(I + K_a(\lambda))^{-1}.$$

The invertibility of $I + K_a(\lambda)$ for $\lambda \in \Omega_M$ is guaranteed by estimate (A.7).

First we prove (A.6). By (A.1) we have

$$\mathcal{E}(t), \partial_t \mathcal{E}(t) = \mathcal{O}(1)_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3}.$$

Thus

$$\chi \mathcal{F}_{t \rightarrow \lambda}(\zeta_a H(t) \mathcal{E}(t) \chi_a) = \mathcal{O}(e^{C_1(\operatorname{Im} \lambda) -})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3},$$

where $C_1 > 0$ is a positive constant, and

$$i\lambda \chi \mathcal{F}_{t \rightarrow \lambda}(\zeta_a H(t) \mathcal{E}(t) \chi_a) = -\mathcal{F}_{t \rightarrow \lambda}(\zeta_a H(t) \partial_t \mathcal{E}(t) \chi_a) = \mathcal{O}(e^{C_1(\operatorname{Im} \lambda) -})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3}.$$

Consequently, we have

$$\chi \mathcal{F}_{t \rightarrow \lambda}(\zeta_a H(t) \mathcal{E}(t) \chi_a) = \mathcal{O}(e^{C_1(\operatorname{Im} \lambda) - \langle \lambda \rangle^{-1}})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3}.$$

In view of (A.5), we have

$$\mathcal{F}_{t \rightarrow \lambda}(\chi W_a(t)) = \mathcal{O}(e^{C_1(\operatorname{Im} \lambda) - \langle \lambda \rangle^{-N}})_{L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3}$$

for any N .

Next is to prove (A.7). We only need to notice that $\chi_b F_a(t)$ and $[\nabla \times \nabla \times, \chi_c] W_a(t)$ both vanish outside a compact set in t . Hence (2.6) is proved.

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