

STABILITY FOR THE INVERSE SOURCE PROBLEM OF THE HELMHOLTZ EQUATION WITH DAMPING*

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Abstract. In this paper, we show for the first time the stability of the inverse source problem for the three-dimensional Helmholtz equation with a variable damping coefficient. The stability estimate consists of the Lipschitz-type data discrepancy and the high wavenumber tail of the source function, where the latter decreases as the upper bound of the wavenumber increases. The stability is shown to have an exponential dependence on the upper bound of the damping coefficient. The analysis employs scattering theory and semiclassical analysis to obtain an analytic domain and an upper bound for the resolvent of the elliptic operator.

Key words. inverse source problem, variable damping, stability, scattering theory, semiclassical analysis

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1. Introduction. The inverse source problem is concerned with determining the unknown source function from measurements of the radiating field away from its support. This problem arises in many scientific and industrial areas such as antenna design and synthesis, biomedical imaging, and photo-acoustic tomography [3]. For example, in medical imaging one uses measurements of the electric or magnetic fields on the surface of the human head generated by neuronal activity of the brain to localize the sources of the activity within the brain, which provide information about the structure of the brain [12]. The inverse source problem is also considered to be a fundamental model in the study of antenna synthesis. In this case, one reconstructs the unknown antenna source from the radiated field, which is measured outside a given region containing the antenna [13]. In general, the uniqueness of the inverse source problem at a fixed energy cannot be guaranteed due to the existence of nonradiating sources [2]. More seriously, inverse problems at a fixed energy are usually lack of

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stability; that is, a small variation in the data leads to huge error in the reconstruction. To deal with these difficulties, the use of multifrequency data has been found to be an effective approach, which helps to regain the uniqueness and achieve increasing stability in numerical experiments [4].

In this paper, we investigate the following three-dimensional damped Helmholtz equation:

$$(1.1) \quad -\Delta u(x, k) - k^2 u(x, k) + ik\sigma(x)u(x, k) = f(x), \quad x \in \mathbb{R}^3,$$

where $k > 0$ is the wavenumber, $\sigma \geq 0$ is known as the damping coefficient, and $f(x) \in L^2(\mathbb{R}^3)$ is the source function. We also assume that f is real-valued. We point out that the solution u to the Helmholtz equation can be considered as a harmonic solution to the time-domain acoustic wave equation $\partial_t^2 U - \Delta U + \sigma(x)\partial_t U = 0$ with damping if U has the form

$$U(x, t) = \Re(u(x)e^{ikt}),$$

where the quantity k is called the pulsation of this harmonic wave [7, 8]. We derive (1.1) by plugging $U(x, t) = \Re(u(x)e^{ikt})$ into the time-domain equation. The Helmholtz equation (1.1) may also be viewed as the Fourier transform of the acoustic wave equation with respect to the time variable via

$$u(x) = \int_0^\infty e^{-ikt} U(x, t) dt.$$

Assume that f has a compact support contained in $B_{\hat{R}} = \{x : x \in \mathbb{R}^3, |x| \leq \hat{R}\}$ with $\hat{R} > 0$ being a constant. Let $R > \hat{R}$ be a constant such that $B_{\hat{R}} \subset B_R = \{x : x \in \mathbb{R}^3, |x| \leq R\}$. Denote the boundary of B_R by ∂B_R . Let $\kappa^2 = k^2 - ik$ with $\Im \kappa > 0$. The Sommerfeld radiation condition is imposed on the scattered field to ensure the well-posedness of the direct scattering problem:

$$(1.2) \quad \lim_{r \rightarrow \infty} r(\partial_r u - i\kappa u) = 0, \quad r = |x|,$$

uniformly in all directions $\hat{x} = x/|x|$. The inverse source problem is to determine the source function $f(x)$ by multiwavenumber boundary measurements $\{u(x, k)|_{\partial B_R}\}$ corresponding to the wavenumber k given in a finite interval.

In practical applications such as antenna synthesis and medical imaging, it is usually the case that the unknown source is embedded in medium with inhomogeneous damping. Therefore, in this work we assume that the damping coefficient is a variable. Specifically, we assume that $\sigma(x) \geq \sigma_0 > 0$ is smooth. We also let $\sigma(x) = 1 + m(x)$ and assume $\text{supp } m(x) \subset B_R$. This means that the damping has a positive lower bound and is homogeneous outside B_R . The existence of damping leads to a spatial decay of the radiating field, which results in poor resolution of the inverse problem. Damping in acoustic waves is small in the air and is large in soft tissues and underwater propagation. It is also a strong factor in electromagnetic detection of soft biological tissues and underground minerals [17]. Uniqueness and numerical methods for inverse source problems without damping have been extensively investigated in [1, 6, 15, 19, 20]. Compared with these works, a main difficulty for the inverse source problem with damping is that now one cannot use plane waves or Dirichlet eigenfunctions of the Laplacian operator as test functions to deduce integral equations which connect the near-field scattering data and unknown source. Moreover, unlike [19], due to the existence of damping, the solution decays exponentially, and thus the far-field pattern

vanishes. Recently, stability estimates for the inverse source problems of various wave equations have been achieved in [5, 11, 17, 21]. However, to the best of our knowledge, the inverse source problem with variable damping coefficients remains wide open to being investigated.

Stability for the inverse source problem with constant damping was obtained in [17]. The proof relied on the explicit Green function to derive bounds for analytic continuation. However, in the current setting with variable damping coefficient the explicit form of the Green function is not available. To overcome this difficulty, the scattering theory and semiclassical analysis in [14, 23] are employed to investigate the meromorphic continuation of the resolvent of the elliptic operator. In particular, we obtain an analytic region and an upper bound of the solution with respect to the complex wavenumber. It should be noticed that the damped wave equation investigated in [23, section 5.3] is defined on torus $\mathbb{T}^3$, which yields the compactness of the resolvent. In contrast, in this work the damped wave equation is defined on the whole space $\mathbb{R}^3$, and thus the compactness is lacking for a general damping. However, thanks to the fact that $\sigma(x)$ is assumed to be a constant outside a compact set in the current setting, perturbation arguments can be applied to regain the compactness of the resolvent. For the inverse problem, we utilized the method developed in [17]. It should be pointed out that the analysis becomes more complicated due to the variable damping coefficient. Specifically, the source function is connected to the scattering data via an observability inequality for the hyperbolic wave equation, which is derived by Carleman estimates. In addition, we use an appropriate rate of time decay of the hyperbolic equation with dissipation to obtain the high wavenumber estimates of the data by Fourier transform. The stability consists of the Lipschitz type of data discrepancy and the high wavenumber tail of the source function. The latter decreases as the wavenumber of the data increases, which implies that the inverse problem is more stable when the higher wavenumber data is used. Moreover, the stability has an exponential dependence on the damping, which suggests that the stability deteriorates as the damping becomes large.

This paper is organized as follows. In section 2 we discuss the direct scattering problem. The resolvent is introduced for the elliptic operator with a variable damping, and an analytic domain and upper bound of the solution are obtained. Section 3 is devoted to the stability analysis of the inverse source problem by using multiwavenumber data. In Appendix A we derive an observability inequality by Carleman estimates.

2. Direct scattering problem. In this section, we study the direct scattering problem. For each $k > 0$, we have the following well-posedness of the direct scattering problem (1.1)–(1.2).

THEOREM 2.1. *For each $k > 0$ the direct scattering problem (1.1)–(1.2) admits a unique solution $u \in H^2(\mathbb{R}^3)$.*

Proof. We first prove the uniqueness of the direct problem. It suffices to show that $u = 0$ if $f = 0$. Since $\text{supp}(1 - \sigma(x)) \subset B_R$, the solution u admits the following Fourier series expansion in polar coordinates for any $|x| = r > R$:

$$(2.1) \quad u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{h_n^{(1)}(\kappa r)}{h_n^{(1)}(\kappa R)} \hat{u}^{(m,n)}(R) Y_n^m(\theta, \phi),$$

where $h_n^{(1)}$ is the spherical Hankel function satisfying

$$h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z), \quad z \in \mathbb{C},$$

Y_n^m are the spherical harmonics of order n , and $\hat{u}^{(m,n)}(R)$ are Fourier coefficients given by

$$\hat{u}^{(m,n)}(R) = \int_{\mathbb{S}^2} u(R, \theta, \phi) \overline{Y_n^m(\theta, \phi)} ds.$$

By [9, Lemmas 2.1 and 2.2] we have $|H_\nu^{(1)}(z)| = |H_\nu^{(1)}(-\bar{z})|$, which yields the following estimate for $z \in \{z \in \mathbb{C} : \Re z \leq 0, \Im z \geq 0\}$:

$$|H_\nu^{(1)}(z)| \leq e^{-\Im z \left(1 - \frac{t^2}{|z|^2}\right)^{1/2}} |H_\nu^{(1)}(t)|,$$

where $\nu \in \mathbb{R}$ and t is any real number with $0 < t \leq |z|$. Thus, since $\Im \kappa > 0$, $\Re \kappa < 0$, we have from (2.1) that for $r \geq R + 1 \geq R$,

$$\begin{aligned} |u(r, \theta, \phi)| &\leq \sum_{n=0}^{\infty} \sum_{m=-n}^n \left| \frac{1}{h_n^{(1)}(\kappa R)} \hat{u}^{(m,n)}(R) Y_n^m(\theta, \phi) \sqrt{\frac{\pi}{2\kappa r}} H_{n+\frac{1}{2}}^{(1)}(\kappa r) \right| \\ &\leq \sqrt{\frac{R+1}{r}} e^{-r\Im \kappa \left(1 - \frac{(R+1)^2}{r^2}\right)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \sum_{m=-n}^n \left| \frac{h_n^{(1)}(|\kappa|(R+1))}{h_n^{(1)}(\kappa R)} \hat{u}^{(m,n)}(R) Y_n^m(\theta, \phi) \right| \\ &\leq C \sqrt{\frac{R+1}{r}} e^{-r\Im \kappa \left(1 - \frac{(R+1)^2}{r^2}\right)^{\frac{1}{2}}} \end{aligned}$$

for some constant C depending on R . An analogous estimate also holds for ∇u . Thus u and ∇u decay exponentially as $r \rightarrow \infty$. Multiplying the equation (1.1) by $\bar{u}$ and integrating by parts over B_r gives

$$\int_{B_r} |\nabla u|^2 - k^2 |u|^2 + ik\sigma |u|^2 dx = \int_{\partial B_r} \frac{\partial u}{\partial \nu} \bar{u} ds(x).$$

Taking the imaginary part of the above identity yields

$$\int_{B_r} k\sigma |u|^2 dx = \Im \int_{\partial B_r} \frac{\partial u}{\partial \nu} \bar{u} ds(x) \rightarrow 0$$

when $r \rightarrow \infty$, which gives $u = 0$. This proves the uniqueness.

For the existence, the scattering problem is equivalent to the problem of the following Lippmann–Schwinger integral equation:

$$u = -ik \int_{\mathbb{R}^3} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} m(y) u(y) dy + \int_{\mathbb{R}^3} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} f(y) dy.$$

Then the existence of the solution follows from a Fredholm alternative argument as in [10] by noticing that the following operator $V : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is compact:

$$V : u \rightarrow -ik \int_{\mathbb{R}^3} \frac{e^{i\kappa|x-y|}}{4\pi|x-y|} m(y) u(y) dy.$$

The proof of existence is completed. Finally, from $u \in L^2(\mathbb{R}^3)$ and the equation (1.1) we have $\Delta u \in L^2(\mathbb{R}^3)$. Therefore, $u \in H^2(\mathbb{R}^3)$. $\square$

In what follows, we investigate the meromorphic continuation of the resolvent of the elliptic operator. Denote the resolvent by

$$R_\sigma(\lambda) = (-\Delta - \lambda^2 + i\lambda\sigma)^{-1} = (-\Delta - \lambda^2 + i\lambda + i\lambda m)^{-1}.$$

Denote a strip in the complex plane by

$$(2.2) \quad \mathcal{R} = [K_0, +\infty) \times (-d, d) \cup (-\infty, -K_0] \times (-d, d),$$

where $K_0, d > 0$ are two positive constants.

The following lemma concerns the meromorphic continuation of the resolvent. In particular, an analytic domain is achieved for the resolvent.

LEMMA 2.2. *The resolvent is meromorphic as a family of operators*

$$R_\sigma(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3).$$

Moreover, for any $K_0 > 0$, there exists $d > 0$ such that $R_\sigma(\lambda)$ is analytic in the strip $\mathcal{R}$.

Proof. Denote $R_0(\lambda) = (-\Delta - \lambda^2 + i\lambda)^{-1}$, which is analytic as a family of operators $R_0(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$. Consider the operator $I + i\lambda m R_0(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$. Since $R_0(\lambda) : L^2(\mathbb{R}^3) \rightarrow H^2(\mathbb{R}^3)$ is a bounded operator and m has compact support, the operator $i\lambda m R_0(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is compact. Moreover, since $R_0(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is analytic with the estimate

$$\|R_0(\lambda)\|_{L^2 \rightarrow L^2} \lesssim \frac{1}{|\sqrt{\lambda^2 - i\lambda}| |\Im \sqrt{\lambda^2 - i\lambda}|},$$

there exists $\lambda \in \mathbb{C}$ with $|\lambda|$ being sufficiently large such that

$$\|i\lambda m R_0(\lambda)\|_{L^2 \rightarrow L^2} \lesssim \frac{|\lambda|}{|\sqrt{\lambda^2 - i\lambda}| |\Im \sqrt{\lambda^2 - i\lambda}|} \ll 1,$$

which implies $I + i\lambda m R_0(\lambda)$ is invertible by Neumann's series argument. Then the operator $(I + i\lambda m R_0(\lambda))^{-1}$ is meromorphic for $\lambda \in \mathbb{C}$ with a countable number of poles by the analytic Fredholm theory. On the other hand, when $I + i\lambda m R_0(\lambda)$ is invertible, the following resolvent identity holds:

$$R_\sigma(\lambda) = R_0(\lambda)(I + i\lambda m R_0(\lambda))^{-1},$$

which gives that $R_\sigma(\lambda) : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is also meromorphic.

In what follows, we show that $R_\sigma(\lambda)$ is analytic for $\lambda \in \mathcal{R}$. According to [14, Theorem 3.33], if the resolvent $R_\sigma(\lambda)$ has a pole at λ , then the direct problem (1.1)–(1.2) admits a nontrivial solution for $f = 0$. Indeed, if R_σ has a pole at λ , then $I - i\lambda m R_0(\lambda)$ is not invertible, which from the Fredholm property gives that it has a nonempty kernel. Let $g = -ikm R_0(\lambda)g$; then $g \in L^2$ and $u := R_0(\lambda)g$ solves

$$\begin{aligned} & (-\Delta - \lambda^2 + i\lambda + i\lambda m)u \\ & = (-\Delta - \lambda^2 + i\lambda)R_0(\lambda)g + i\lambda m R_0(\lambda)g = g + i\lambda m R_0(\lambda)g = 0. \end{aligned}$$

On the other hand, we see from the proof of Theorem 2.1 that the direct problem only admits a trivial solution for $\lambda \in \mathcal{R}$ with d being sufficiently small. Consequently, $R_\sigma(\lambda)$ has no pole and thus is analytic for $\lambda \in \mathcal{R}$. $\square$

Next, we consider the resolvent estimates. To achieve this goal, we employ the theory of semiclassical analysis. Letting $z = h^2 k^2$, we see that the equation (1.1) becomes

$$(2.3) \quad -h^2 \Delta u + i\sqrt{z} h \sigma u - zu = h^2 f.$$

Denote the elliptic operator by

$$P(z, h) = -h^2 \Delta + i\sqrt{z} h \sigma(x) - z.$$

If $P(z, h)$ is invertible, we denote its inverse operator by

$$\tilde{R}_\sigma(z, h) = P(z, h)^{-1}.$$

DEFINITION 2.3. Let $S^m(\mathbb{R}^3)$ be the space of all functions $a(x, \xi)$ where $x, \xi \in \mathbb{R}^3$ such that $a(x, \xi) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^m$$

with $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. If $a \in S^m(\mathbb{R}^3)$, we define the Weyl quantization to be the operator $A = a^w(x, hD)$ acting on $u \in \mathcal{S}$ as follows:

$$Au(x) = (2\pi h)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{\frac{i}{h}(x-y) \cdot \xi} a\left(\frac{x+y}{2}, \xi\right) u(y) d\xi dy.$$

The following two propositions concern the microlocal defect measure and its support.

PROPOSITION 2.4 ([23, Theorem 5.2]). Let $\{u(h)\}, 0 < h \leq h_0$ be a family of functions in $L^2(\mathbb{R}^3)$ satisfying $\|u(h)\|_{L^2(\mathbb{R}^3)} = 1$. Then there exist a Radon measure on $\mathbb{R}^3 \times \mathbb{R}^3$ and a sequence $h_j \rightarrow 0$ such that

$$\langle a^w(x, h_j D) u(h_j), u(h_j) \rangle \rightarrow \int_{\mathbb{R}^3 \times \mathbb{R}^3} a(x, \xi) d\mu$$

for all symbols $a \in S^0(\mathbb{R}^3)$. The Radon measure μ is called a microlocal defect measure associated with the family $\{u(h)\}, 0 < h \leq h_0$.

PROPOSITION 2.5 ([23, Theorem 5.3]). Let $p \in S^m(\mathbb{R}^3)$ be real and

$$|p| \geq \gamma \langle \xi \rangle^m \quad \text{for } |\xi| \geq C,$$

for constants $C, \gamma > 0$. Write $P(h) = p^w$. Suppose that $u(h)$ satisfies

$$\|P(h)u(h)\|_{L^2(\mathbb{R}^3)} = o(1) \text{ as } h \rightarrow 0, \quad \|u\|_{L^2(\mathbb{R}^3)} = 1.$$

Then if μ is any microlocal defect measure associated with the family $\{u(h)\}, 0 < h \leq h_0$, it holds that

$$\text{supp } \mu \subset p^{-1}(0).$$

According to [23], the following two lemmas hold. We present the proofs with adaptations to the current setting for the reader's convenience.

LEMMA 2.6. There exist constants α, C , and $h_0 > 0$ such that

$$\|\tilde{R}_\sigma(z, h)\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq \frac{C}{h}$$

for

$$(2.4) \quad |\Im z| \leq \alpha h, |z - 1| \leq \alpha, \quad 0 < h \leq h_0.$$

Proof. It suffices to prove that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{h} \|P(z, h)u\|_{L^2(\mathbb{R}^3)}$$

for all $u \in L^2$ if z and h satisfy the assumption (2.4).

We prove by an argument of contradiction. If the result were not true, then there would exist $z_m \in \mathbb{C}$, $0 < h_m < \frac{1}{m}$, and functions $u_m \in L^2(\mathbb{R}^3)$ such that

$$\|P(z_m, h_m)u_m\|_{L^2(\mathbb{R}^3)} \leq \frac{h_m}{m} \|u_m\|_{L^2(\mathbb{R}^3)}, \quad |\Im z_m| \leq \frac{h_m}{m}, \quad |z_m - 1| \leq \frac{1}{m}.$$

We normalize u_m such that $\|u_m\|_{L^2(\mathbb{R}^3)} = 1$. Then

$$(2.5) \quad P(z_m, h_m)u_m = o(h_m), \quad z_m \rightarrow 1, \quad \Im z_m = o(h_m).$$

Let μ be the microlocal defect measure associated with the sequence $\{u_m\}_{m=1}^\infty$. It is straightforward to verify that $P_0(h) = p^\omega$ with $P_0(h) = -h^2\Delta - 1$ and $p := |\xi|^2 - 1$. Then we have

$$\|P_0(h_m)u_m\|_{L^2(\mathbb{R}^3)} \leq \|P(z_m, h_m)u_m\|_{L^2(\mathbb{R}^3)} + \|(P(z_m, h_m) - P_0(h_m))u_m\|_{L^2(\mathbb{R}^3)} = o(1)$$

as $m \rightarrow \infty$. Therefore, Proposition 2.5 implies that

$$\text{supp}(\mu) \subset p^{-1}(0) = \{\xi : |\xi| = 1\}.$$

As $\|u_m\|_{L^2(\mathbb{R}^3)} = 1$, letting $a = 1 \in S^0(\mathbb{R}^3)$ in Proposition 2.4 gives

$$(2.6) \quad \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mu = 1.$$

In what follows we derive a contradiction to the identity (2.6). Let $P_m = P(z_m, h_m)$. Then we have

$$P_m = -h_m^2\Delta + i\sqrt{z_m}h_m\sigma(x) - z_m$$

and the adjoint of P_m as follows:

$$P_m^* = -h_m^2\Delta - i\sqrt{\bar{z}_m}h_m\sigma(x) - \bar{z}_m,$$

which yields

$$P_m - P_m^* = i(\sqrt{z_m} + \sqrt{\bar{z}_m})h_m\sigma(x) - z_m + \bar{z}_m = 2ih_m\sigma(x) + o(h_m)$$

since by (2.5), $\sqrt{z_m} = 1 + o(1)$ and $-z_m + \bar{z}_m = -2i\Im(z_m) = o(h_m)$.

Now we let $b \in S^{-2}$ and introduce the operator $B_m = b^w(x, h_mD)$. Then $B_m = B_m^*$. From (2.5) a direct calculation yields

$$\begin{aligned} o(h_m) &= 2i\Im(B_m P_m u_m, u_m) = \langle B_m P_m u_m, u_m \rangle - \langle u_m, B_m P_m u_m \rangle \\ (2.7) \quad &= \langle (B_m P_m - P_m^* B_m)u_m, u_m \rangle \\ &= \langle [B_m, P_m]u_m, u_m \rangle + \langle (P_m - P_m^*)B_m u_m, u_m \rangle \\ &= -h_m^2 \langle [B_m, \Delta]u_m, u_m \rangle + i\sqrt{z_m}h_m \langle [B_m, \sigma]u_m, u_m \rangle + \langle (P_m - P_m^*)B_m u_m, u_m \rangle \\ &= -h_m^2 \langle [B_m, \Delta]u_m, u_m \rangle + ih_m \langle [B_m, \sigma]u_m, u_m \rangle + 2ih_m \langle \sigma B_m u_m, u_m \rangle + o(h_m) \\ &= -h_m^2 \langle [B_m, \Delta]u_m, u_m \rangle + ih_m \langle (B_m \sigma + \sigma B_m)u_m, u_m \rangle + o(h_m). \end{aligned}$$

Then applying [23, Theorems 4.18 and 4.24] gives

$$(2.8) \quad -h_m^2[B_m, \Delta] = [B_m, P_0(h_m)] = \frac{h_m}{i}\{b, p\}^w + o(h_m),$$

$$(2.9) \quad B_m \sigma + \sigma B_m = 2(\sigma b)^\omega + o(h_m),$$

where the Poisson bracket $\{, \}$ is defined as $\{b, p\} = \nabla_\xi b \cdot \nabla_x p - \nabla_x b \cdot \nabla_\xi p$. Inserting (2.8)–(2.9) into (2.7) gives

$$\begin{aligned} o(h_m) &= \frac{h_m}{i} \langle \{b, p\}^w u_m, u_m \rangle \\ &\quad + 2h_m i \langle (\sigma b)^\omega u_m, u_m \rangle + o(h_m). \end{aligned}$$

Dividing the above identity by h_m/i and letting $h_m \rightarrow 0$ gives

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \{p, b\} + 2\sigma b \, d\mu = 0.$$

Next, we choose b such that $\{p, b\} + 2\sigma b > 0$ on $\text{supp } \mu$, which implies $\int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mu = 0$ and leads to a contradiction to (2.6). Recalling $\sigma \geq \sigma_0 > 0$, for each initial point $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|\xi| = 1$, there exists $T > 0$ such that

$$\langle \sigma \rangle_T = \frac{1}{T} \int_0^T \sigma(x + t\xi) dt > 0.$$

Define

$$c(x, \xi) = \frac{1}{T} \int_0^T (T - t) \sigma(x + \xi t) dt,$$

where $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $|\xi| = 1$. Consequently, we have

$$\begin{aligned} \langle \xi, \nabla_x c \rangle &= \frac{1}{T} \int_0^T (T - t) \langle \xi, \nabla \sigma(x + t\xi) \rangle dt \\ &= \frac{1}{T} \int_0^T (T - t) \frac{d}{dt} \sigma(x + t\xi) dt \\ &= \frac{1}{T} \int_0^T \sigma(x + t\xi) dt - \sigma(x) \\ &= \langle \sigma \rangle_T - \sigma. \end{aligned}$$

Now we let $b(x, \xi) = e^{c(x, \xi)} \chi(p) = e^{c(x, \xi)} \chi(|\xi|^2 - 1)$, where $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi = 1$ near 0. Recalling that the assumption $\sigma - 1 \in C_c^\infty(\mathbb{R}^3)$ ensures that any order derivative of σ is bounded, it is easy to verify $b \in S^{-2}$. Then

$$\langle \xi, \nabla_x b \rangle = e^c \langle \xi, \nabla_x c \rangle \chi(p) = e^c \langle \sigma \rangle_T \chi(p) - e^c \sigma \chi(p)$$

since $\{p, \chi(p)\} = 0$. Thus, we have

$$\{p, b\} + 2\sigma b = 2\langle \xi, \nabla_x b \rangle + 2\sigma b = 2e^c \langle \sigma \rangle_T \chi(p) > 0 \quad \text{on } p^{-1}(0).$$

The proof is completed. $\square$

The following lemma concerns the resolvent estimates.

LEMMA 2.7. *The following resolvent estimates hold:*

$$\|R_\sigma(k)\|_{L^2(\mathbb{R}^3) \rightarrow H^j(\mathbb{R}^3)} \leq C|k|^{j-1}, \quad k \in \mathcal{R}, \quad j = 0, 1, 2.$$

Proof. By letting $z = h^2 k^2$ and by (2.3) we have

$$R_\sigma(k) = h^2 \tilde{R}_\sigma(z, h).$$

Define the semiclassical norm of H^2 by $\|v\|_{H_h^2(\mathbb{R}^3)} = h^2 \|\Delta v\|_{L^2(\mathbb{R}^3)} + \|v\|_{L^2(\mathbb{R}^3)}$. For the case where $j = 2$, we have

$$\|h^2 \Delta \tilde{R}_\sigma(z, h) f\|_{L^2(\mathbb{R}^3)} \leq \|(z + i\sqrt{z}h\sigma) \tilde{R}_\sigma(z, h) f + f\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{h} \|f\|_{L^2(\mathbb{R}^3)},$$

where the last inequality holds by Lemma 2.6. Thus,

$$(2.10) \quad \|\tilde{R}_\sigma(z, h) f\|_{H_h^2(\mathbb{R}^3)} \leq \frac{C}{h} \|f\|_{L^2(\mathbb{R}^3)}.$$

Now put

$$z = h^2 k^2, \quad k = k_1 + ik_2, \quad h = \frac{1}{k_1},$$

such that

$$|\Im z| \leq \alpha h, \quad |z - 1| \leq \alpha \quad \text{if} \quad |k_2| \leq \gamma, \quad |k_1| > \frac{1}{\gamma}$$

for some sufficiently small γ . Then inequalities in (2.4) hold. Therefore, for $|k_2| \leq \gamma$ and $|k_1| \geq 1/\gamma$, (2.10) implies

$$(2.11) \quad \|R_\sigma(k) f\|_{H^2(\mathbb{R}^3)} \leq C|k| \|f\|_{L^2(\mathbb{R}^3)}.$$

Thus, if $\mathcal{R}$ is as defined in (2.2) with K_0 being sufficiently large and d being sufficiently small, the estimate (2.11) holds for $k \in \mathcal{R}$.

Using a similar rescaling argument, we also have by Lemma 2.6

$$\|R_\sigma(k) f\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{|k|} \|f\|_{L^2}, \quad k \in \mathcal{R}.$$

The remaining case of $j = 1$ follows from interpolation. The proof is completed. $\square$

3. Inverse scattering problem. In this section, we discuss the stability of the inverse problem. We introduce the boundary measurements as follows:

$$(3.1) \quad \|u(\cdot, k)\|_{\partial B_R}^2 := \int_{\partial B_R} \left(k^2 |u(x, k)|^2 + |\nabla u(x, k)|^2 \right) ds(x).$$

In fact, for the boundary measurements, we only need to measure the Dirichlet data u on the boundary ∂B_R , and the gradient ∇u on ∂B_R can be computed from u on ∂B_R . To illustrate this, note that the normal derivative $\partial_\nu u$ on ∂B_R can be computed via the Dirichlet-to-Neumann map. Specifically, as the solution u admits the following Fourier series expansion in polar coordinates for any $|x| = r > R$:

$$(3.2) \quad u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \frac{h_n^{(1)}(\kappa r)}{h_n^{(1)}(\kappa R)} \hat{u}^{(m, n)}(R) Y_n^m(\theta, \phi),$$

we can compute $\partial_\nu u$ by taking the derivative of (3.2), which yields

$$\partial_\nu u(R, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \kappa \frac{(h_n^{(1)})'(\kappa R)}{h_n^{(1)}(\kappa R)} \hat{u}^{(m,n)}(R) Y_n^m(\theta, \phi).$$

We can also compute the tangential derivative of u on ∂B_R from u on ∂B_R , which gives ∇u . Thus, we may only use Dirichlet data on the measurement boundary.

Hereafter, the notation $a \lesssim b$ stands for $a \leq Cb$, where $C > 0$ is a generic constant, which may change step by step in the proofs.

The following lemma connects the unknown source and the boundary measurements.

LEMMA 3.1. *Let u be the solution to the direct scattering problem (1.1)–(1.2). Then there exists a constant $C > 0$ such that*

$$\|f\|_{L^2(B_R)}^2 \lesssim e^{C\|\sigma\|_{L^\infty}^2} \int_0^{+\infty} \|u(x, k)\|_{\partial B_R}^2 dk.$$

Proof. Define $U(x, t) = 0$ in the hyperbolic equation (A.1) when $t < 0$, and define the Fourier transform as follows:

$$\widehat{U}(x, k) = \int_0^{+\infty} U(x, t) e^{-ikt} dt.$$

Since $\sigma \geq \sigma_0 > 0$ is positive, we have from known results in [18] that

$$\|U(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq C(f)(1+t)^{-3/2}.$$

Hence, the Fourier transform is well defined. Furthermore, using integration by parts and the finite speed of propagation, we obtain that $\widehat{U}(x, k)$ satisfies the direct scattering problem (1.1)–(1.2), which gives $\widehat{U}(x, k) = u(x, k)$; that is, the hyperbolic equation (A.1) is converted to the Helmholtz equation (1.1) through the Fourier transform.

From the observability inequality in (A.1) and noting that $\partial_t \widehat{U}(x, t) = iku(x, k)$, we have by using Parseval's identity that

$$\begin{aligned} \|f\|_{L^2(B_R)}^2 &\leq C e^{C\|\sigma\|_{L^\infty}^2} (\|\partial_t U\|_{L^2(\partial B_R \times (0, T))}^2 + \|\nabla U\|_{L^2(\partial B_R \times (0, T))}^2) \\ &\leq C e^{C\|\sigma\|_{L^\infty}^2} \int_0^\infty \int_{\partial B_R} (k^2 |u(x, k)|^2 + |\nabla u(x, k)|^2) ds(x) dk, \end{aligned}$$

which completes the proof. $\square$

The following lemma gives an analytic continuation principle (cf. [21, Lemma A.1]).

LEMMA 3.2. *Let $p(z)$ be analytic in the infinite rectangular slab $\widetilde{\mathcal{R}} := \{z \in \mathcal{R} : \Re z > 0\}$, and continuous in $\overline{\widetilde{\mathcal{R}}}$ satisfying*

$$\begin{cases} |p(z)| \leq \epsilon, & z \in (K_0, K], \\ |p(z)| \leq M, & z \in \widetilde{\mathcal{R}}, \end{cases}$$

where K_0, K, ϵ , and M are positive constants. Then there exists a function $\mu(z)$ with $z \in (K, +\infty)$ satisfying

$$(3.3) \quad \mu(z) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - z)},$$

where $a = K - K_0$, such that

$$(3.4) \quad |p(z)| \leq M\epsilon^{\mu(z)} \quad \forall z \in (K, +\infty).$$

We introduce the following multiwavenumber data:

$$I(k) = \int_{K_0}^k \|u(x, \omega)\|_{\partial B_R}^2 d\omega, \quad k > 0,$$

where K_0 is specified in (2.2).

The following lemma provides an analytic continuation of the data from low wavenumbers to higher wavenumbers.

LEMMA 3.3. Assume that f is real-valued with $\|f\|_{L^2(B_R)} \leq Q$. We have

$$|I(k)| \lesssim Q^2 e^{ck} \epsilon^{2\mu(k)} \quad \forall k \in (K, +\infty),$$

where

$$\epsilon^2 = \int_{K_0}^K \|u(x, k)\|_{\partial B_R}^2 dk, \quad \mu(k) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - k)},$$

and $a = K - K_0$. Here K_0 is specified in (2.2).

Proof. Extend $I(k)$ from $k > 0$ to $k \in \tilde{\mathcal{R}}$ as follows:

$$\tilde{I}(k) = \int_{K_0}^k \int_{\partial B_R} \left(\omega^2 u(x, \omega) u(x, -\omega) + \nabla u(x, \omega) \cdot \nabla u(x, -\omega) \right) ds(x) d\omega,$$

where $k \in \tilde{\mathcal{R}}$. Here the integral is taken over the line segment from K_0 to $k \in \tilde{\mathcal{R}}$. Since f is real-valued, we have $\overline{u(x, k)} = u(x, -k)$ for $k \in \mathbb{R}$, which gives

$$\tilde{I}(k) = I(k), \quad k > 0.$$

It follows from Lemma 2.7 that

$$|\tilde{I}(k)| \lesssim Q^2 |k|^2,$$

which gives

$$e^{-c|k|} |\tilde{I}(k)| \lesssim Q^2, \quad k \in \tilde{\mathcal{R}},$$

where $c > 0$ is any positive constant. Notice that for $k \in \tilde{\mathcal{R}}$ we have $|\Im k| < d$, which implies

$$e^{-c\Re k} |\tilde{I}(k)| \lesssim e^{-c|k|} |\tilde{I}(k)| \lesssim Q^2, \quad k \in \tilde{\mathcal{R}}.$$

An application of Lemma 3.2 leads to

$$|e^{-ck} I(k)| \lesssim Q^2 \epsilon^{2\mu(k)} \quad \forall k \in (K, +\infty),$$

where

$$\mu(k) \geq \frac{64ad}{3\pi^2(a^2 + 4d^2)} e^{\frac{\pi}{2d}(\frac{a}{2} - k)}$$

and $a = K - K_0$, which completes the proof. $\square$

The following result provides an estimate of the data for high wavenumbers.

LEMMA 3.4. Let $f \in H^2(B_{\hat{R}})$ and $\|f\|_{H^2(B_{\hat{R}})} \leq Q$. Then the following estimate holds:

$$\int_k^\infty \|u(x, \omega)\|_{\partial B_R}^2 d\omega \lesssim \frac{1}{k^2} \|f\|_{H^2(B_{\hat{R}})}^2.$$

Proof. Let Ω be a bounded domain in $\mathbb{R}^3$. According to [22, Proposition 1.1], we have

$$(3.5) \quad \|U(\cdot, t)\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^1(B_{\hat{R}})}^2, \quad \|\Delta U(\cdot, t)\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-2} \|f\|_{H^1(B_{\hat{R}})}^2,$$

and

$$(3.6) \quad \|\partial_t^2 U\|_{L^2(\Omega)}^2 + \|\nabla \partial_t U\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-3} \|f\|_{H^1(B_{\hat{R}})}^2.$$

Therefore, it follows from interior elliptic regularity theory and (3.5) that

$$(3.7) \quad \|U(\cdot, t)\|_{H^2(B_R)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^1(B_{\hat{R}})}^2.$$

Let Ω_0 be an open domain such that $\partial B_R \subset \Omega_0$ and $\Omega_0 \cap \text{supp}(f) = \emptyset$. It follows from the estimates (3.5), (3.6) and the equation (A.1) that

$$\|\partial_t U(\cdot, t)\|_{L^2(\Omega_0)}^2 \lesssim (1+t)^{-2} \|f\|_{H^1(B_{\hat{R}})}^2.$$

In view of (3.6) again we have $\|\nabla \partial_t U\|_{L^2(\Omega_0)}^2 \lesssim (1+t)^{-3} \|f\|_{H^1(B_{\hat{R}})}^2$, which combined with the above estimate yields

$$\|\partial_t U(\cdot, t)\|_{H^1(\Omega_0)}^2 \lesssim (1+t)^{-2} \|f\|_{H^1(B_{\hat{R}})}^2.$$

On the other hand, from (3.7) we obtain

$$\|\nabla U(\cdot, t)\|_{L^2(\partial B_R)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^1(B_{\hat{R}})}^2.$$

In summary, we have

$$(3.8) \quad \|\nabla U(\cdot, t)\|_{L^2(\partial B_R)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^1(B_{\hat{R}})}^2, \quad \|\partial_t U(\cdot, t)\|_{L^2(\partial B_R)}^2 \lesssim (1+t)^{-2} \|f\|_{H^1(B_{\hat{R}})}^2.$$

Letting $\tilde{U} = \partial_t U$, we see that $\tilde{U}$ satisfies the following initial-valued acoustic wave equation:

$$\begin{cases} \partial_t^2 \tilde{U} - \Delta \tilde{U} + \sigma(x) \partial_t \tilde{U} = 0, & (x, t) \in B_R \times (0, +\infty), \\ \tilde{U}(x, 0) = f, \quad \partial_t \tilde{U}(x, 0) = -\sigma f, & x \in B_R. \end{cases}$$

Then by applying [22, Proposition 1.1] to $\tilde{U}$, we have

$$(3.9) \quad \|\tilde{U}(\cdot, t)\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^2(B_{\hat{R}})}^2, \quad \|\Delta \tilde{U}(\cdot, t)\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-2} \|f\|_{H^2(B_{\hat{R}})}^2,$$

and

$$\|\partial_t^2 \tilde{U}\|_{L^2(\Omega)}^2 + \|\nabla \partial_t \tilde{U}\|_{L^2(\Omega)}^2 \lesssim (1+t)^{-3} \|f\|_{H^2(B_{\hat{R}})}^2.$$

Notice that the H^2 norm of f appears on the right-hand side of the estimates since now the initial condition of $\tilde{U}$ is $\tilde{U}(x, 0) = f$, $\partial_t \tilde{U}(x, 0) = -\sigma f$. Therefore, it follows from interior elliptic regularity theory and (3.9) that

$$\|\tilde{U}(\cdot, t)\|_{H^2(B_R)}^2 \lesssim (1+t)^{-7/4} \|f\|_{H^2(B_{\hat{R}})}^2.$$

Then, following a similar argument for the derivation of (3.8), we obtain the following estimates of $\partial_t U$:

$$(3.10) \quad \begin{aligned} \|\nabla \partial_t U(\cdot, t)\|_{L^2(\partial B_R)}^2 &\lesssim (1+t)^{-7/4} \|f\|_{H^2(B_{\hat{R}})}^2, \quad \|\partial_t^2 U(\cdot, t)\|_{L^2(\partial B_R)}^2 \\ &\lesssim (1+t)^{-2} \|f\|_{H^2(B_{\hat{R}})}^2. \end{aligned}$$

We have by Parseval's identity and (3.10)

$$\begin{aligned} \int_{|\omega|>k} \omega^2 \|u(\cdot, \omega)\|_{L^2(\partial B_R)}^2 d\omega &\leq k^{-2} \int_{|\omega|>k} \omega^4 \|u(\cdot, \omega)\|_{L^2(\partial B_R)}^2 d\omega \\ &\leq k^{-2} \int_{\mathbb{R}} \omega^4 \|u(\cdot, \omega)\|_{L^2(\partial B_R)}^2 d\omega \\ &= k^{-2} \int_{\mathbb{R}} \|\partial_t^2 U(\cdot, t)\|_{L^2(\partial B_R)}^2 dt \\ &\lesssim k^{-2} \|f\|_{H^2(B_{\hat{R}})}^2 \int_0^\infty (1+t)^{-2} dt \\ &\lesssim k^{-2} \|f\|_{H^2(B_{\hat{R}})}^2, \end{aligned}$$

and

$$\begin{aligned} \int_{|\omega|>k} \|\nabla u(\cdot, \omega)\|_{L^2(\partial B_R)}^2 d\omega &\lesssim k^{-2} \int_{\mathbb{R}} \|\partial_t \nabla U(\cdot, t)\|_{L^2(\partial B_R)}^2 dt \\ &\lesssim k^{-2} \|f\|_{H^2(B_{\hat{R}})}^2 \int_0^\infty (1+t)^{-7/4} dt \\ &\lesssim k^{-2} \|f\|_{H^2(B_{\hat{R}})}^2. \end{aligned}$$

The proof is finished by noticing that $|u(x, \omega)| = |u(x, -\omega)|$ for $\omega > 0$. $\square$

The following stability estimate is the main result of this paper. The proof utilizes an argument of analytic continuation as in [21].

THEOREM 3.5. *Let $f \in H^2(B_R)$ with $\|f\|_{H^2(B_R)} \leq Q$. There exists a positive constant $C > 0$ such that the following stability estimate holds:*

$$(3.11) \quad \|f\|_{L^2(B_R)}^2 \lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + \frac{Q^2}{K \ln |\ln \epsilon|} \right),$$

where

$$\epsilon^2 := \int_0^K \|u(x, k)\|_{\partial B_R}^2 dk$$

and the boundary measurement $\|u(x, k)\|_{\partial B_R}^2$ is defined as in (3.1).

Proof. We can assume that $\epsilon \leq e^{-1}$; otherwise the estimate is obvious.

Let $L > 0$ be a large constant to be determined later. By applying the analytic continuation in Lemma 3.3 we have

$$\begin{aligned} I(k) &\lesssim Q^2 e^{c|k|} \epsilon^{\mu(k)} \\ &\lesssim Q^2 \exp \left\{ ck - \frac{c_2 a}{a^2 + c_3} e^{c_1(\frac{a}{2} - k)} |\ln \epsilon| \right\} \\ &\lesssim Q^2 \exp \left\{ -\frac{c_2 a}{a^2 + c_3} e^{c_1(\frac{a}{2} - k)} |\ln \epsilon| \left(1 - \frac{c_4 k(a^2 + c_3)}{a} e^{c_1(k - \frac{a}{2})} |\ln \epsilon|^{-1} \right) \right\} \\ &\lesssim Q^2 \exp \left\{ -\frac{c_2 a}{a^2 + c_3} e^{c_1(\frac{a}{2} - L)} |\ln \epsilon| \left(1 - \frac{c_4 L(a^2 + c_3)}{a} e^{c_1(L - \frac{a}{2})} |\ln \epsilon|^{-1} \right) \right\} \\ &\lesssim Q^2 \exp \left\{ -b_0 e^{-c_1 L} |\ln \epsilon| (1 - b_1 L e^{c_1 L} |\ln \epsilon|^{-1}) \right\} \end{aligned}$$

for $K < k \leq L$, where $c, c_i, i = 1, 2, 3$, and b_0, b_1 are constants. Let

$$L = \begin{cases} \left\lceil \frac{1}{2c_1} \ln |\ln \epsilon| \right\rceil, & K \leq \frac{1}{2c_1} \ln |\ln \epsilon|, \\ K, & K > \frac{1}{2c_1} \ln |\ln \epsilon|. \end{cases}$$

If $K \leq \frac{1}{2c_1} \ln |\ln \epsilon|$, we obtain for sufficiently small ϵ that

$$\begin{aligned} I(k) &\lesssim Q^2 \exp \left\{ -b_0 e^{-c_1 L} |\ln \epsilon| (1 - b_1 L e^{c_1 L} |\ln \epsilon|^{-1}) \right\} \\ &\lesssim Q^2 \exp \left\{ -\frac{1}{2} b_0 e^{-c_1 L} |\ln \epsilon| \right\} \end{aligned}$$

for $K < k \leq L$. Noting that $e^{-x} \leq \frac{1}{x}$ for $x > 0$, we have

$$I(L) \lesssim Q^2 e^{c_1 L} |\ln \epsilon|^{-1}.$$

Taking $L = \frac{1}{2c_1} \ln |\ln \epsilon|$ and combining the above estimates and Lemmas 3.1 and 3.4, we get

$$\begin{aligned} \|f\|_{L^2(B_R)}^2 &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + I(L) + \int_L^\infty \|u(x, k)\|_{\partial B_R}^2 dk \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + Q^2 e^{c_1 L} |\ln \epsilon|^{-1} + \frac{Q^2}{L^2} \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + Q^2 \left(|\ln \epsilon|^{\frac{1}{2}} |\ln \epsilon|^{-1} + (\ln |\ln \epsilon|)^{-2} \right) \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + Q^2 \left(|\ln \epsilon|^{-\frac{1}{2}} + (\ln |\ln \epsilon|)^{-2} \right) \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + Q^2 (\ln |\ln \epsilon|)^{-2} \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + \frac{Q^2}{K \ln |\ln \epsilon|} \right), \end{aligned}$$

where we have used $|\ln \epsilon|^{1/2} \geq \ln |\ln \epsilon|$ for sufficiently small ϵ .

If $K > \frac{1}{2c_1} \ln |\ln \epsilon|$, we have from Lemma 3.4 that

$$\begin{aligned} \|f\|_{L^2(B_R)}^2 &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + \int_K^\infty \|u(x, k)\|_{\partial B_R}^2 dk \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + \frac{Q^2}{K^2} \right) \\ &\lesssim e^{C\|\sigma\|_{L^\infty}^2} \left(\epsilon^2 + \frac{Q^2}{K \ln |\ln \epsilon|} \right), \end{aligned}$$

which completes the proof. $\square$

The stability (3.11) consists of two parts: the data discrepancy and the high wavenumber tail. The former is of the Lipschitz type. The latter decreases as K increases, which makes the problem have an almost Lipschitz stability. The result reveals that the problem becomes more stable when higher wavenumber data is used. The stability also exhibits an exponential dependence on the upper bound of the damping. As the damping increases, the stability deteriorates.

4. Conclusion. We have presented a stability result for the inverse source problem of time-harmonic acoustic waves in inhomogeneous background damping. An increasing stability is derived by multiwavenumber boundary measurements, which consists of the data discrepancy and the high wavenumber tail of the source function. The result shows that the ill-posedness of the inverse source problem decreases as the wavenumber increases for the data. In addition, the stability has an exponential dependence on the damping, which deteriorates as the damping increases. The analysis employs scattering theory and semiclassical analysis to achieve an analytic domain and explicit bounds of the data for analytic continuation. A possible continuation is to consider the inverse source problem in an inhomogeneous medium with damping. In this case, we may derive a similar observability inequality by using Carleman estimates in [16]. Moreover, the medium should be assumed to be nontrapping in order to apply the scattering theory. As a more challenging problem, we hope to study the inverse problem of determining the damping coefficient.

Appendix A. The observability inequality for the wave equation. In this section, we derive an observability inequality for the hyperbolic equation with dissipation.

Consider the initial-valued acoustic wave equation

$$(A.1) \quad \begin{cases} \partial_t^2 U - \Delta U + \sigma(x) \partial_t U = 0, & (x, t) \in B_R \times (0, +\infty), \\ U(x, 0) = 0, \quad \partial_t U(x, 0) = f, & x \in B_R. \end{cases}$$

The following theorem provides an exact observability bound for the above equation.

THEOREM A.1. *Let the observation time $4(2R+1) < T < 5(2R+1)$. Then there exists a constant C depending on the domain B_R such that*

$$(A.2) \quad \|f\|_{L^2(B_R)}^2 \leq C e^{C\|\sigma\|_{L^\infty}^2} (\|\partial_t U\|_{L^2(\partial B_R \times (0, T))}^2 + \|\nabla U\|_{L^2(\partial B_R \times (0, T))}^2)$$

for all $U \in H^2(B_R \times (0, +\infty))$ solving (A.1).

The proof of Theorem A.1 is motivated by that of Theorem 3.1 in [17] for constant dampings. Before the formal proof, we introduce the energies

$$E(t) := \frac{1}{2} \int_{B_R} (|\partial_t U(x, t)|^2 + |\nabla U(x, t)|^2 + |U(x, t)|^2) dx,$$

$$E_0(t) := \frac{1}{2} \int_{B_R} (|\partial_t U(x, t)|^2 + |\nabla U(x, t)|^2) dx,$$

and denote

$$F^2 := \frac{1}{2} \int_{\partial B_R \times (0, T)} (|\partial_t U(x, t)|^2 + |\nabla U(x, t)|^2 + U^2(x, t)) ds(x) dt.$$

LEMMA A.2. *Let U be a solution of the forward problem (A.1) with $f \in H^1(B_R)$ and $\text{supp } f \subset B_R$. Let $0 \leq t_1 < t_2 \leq T$.*

Then

$$(A.3) \quad E(t_2) \leq e^{4(t_2 - t_1)^2} (2E(t_1) + F^2)$$

and

$$(A.4) \quad E(t_1) \leq 2e^{(2\|\sigma\|_{L^\infty} + 4(t_2 - t_1))(t_2 - t_1)} (E(t_2) + F^2).$$

Proof. Let $\theta \geq 0$ be a constant. Multiplying the equation (A.1) by $(\partial_t U)e^{\theta t}$ and integrating over $B_R \times (t_1, t_2)$ yields

$$\int_{B_R \times (t_1, t_2)} \left(\frac{1}{2} \partial_t (\partial_t U)^2 - \Delta U \partial_t U + \sigma (\partial_t U)^2 \right) e^{\theta t} dx dt = 0.$$

Integrating $\Delta U \partial_t U$ by parts over B_R and using that $\nabla U \cdot \nabla \partial_t U = \frac{1}{2} \partial_t (|\nabla U|^2)$, we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} (\partial_t E_0(t)) e^{\theta t} dt + \int_{B_R \times (t_1, t_2)} \sigma (\partial_t U)^2 e^{\theta t} dx dt \\ & - \int_{\partial B_R \times (t_1, t_2)} \partial_\nu U \partial_t U e^{\theta t} ds(x) dt = 0. \end{aligned}$$

Hence,

$$\begin{aligned} E_0(t_2) e^{\theta t_2} - E_0(t_1) e^{\theta t_1} &= \int_{B_R \times (t_1, t_2)} \left(\frac{\theta}{2} ((\partial_t U)^2 + |\nabla U|^2) - \sigma (\partial_t U)^2 \right) e^{\theta t} dx dt \\ &+ \int_{\partial B_R \times (t_1, t_2)} \partial_\nu U \partial_t U e^{\theta t} ds(x) dt. \end{aligned}$$

Letting $\theta = 0$ and using the elementary inequality $2AB \leq A^2 + B^2$, we derive

$$\begin{aligned} E_0(t_2) &\leq E_0(t_1) + \int_{B_R \times (t_1, t_2)} (-\sigma) (\partial_t U)^2 dx dt \\ &+ \frac{1}{2} \int_{\partial B_R \times (t_1, t_2)} ((\partial_\nu U)^2 + (\partial_t U)^2) ds(x) dt \\ (A.5) \quad &\leq E_0(t_1) + F^2 \end{aligned}$$

noticing that $\sigma = \sigma(x) > 0, x \in \mathbb{R}^n$.

Similarly, letting $\theta = 2\|\sigma\|_{L^\infty}$, we derive

$$\begin{aligned} & E_0(t_1)e^{2\|\sigma\|_{L^\infty}t_1} \\ & \leq E_0(t_2)e^{2\|\sigma\|_{L^\infty}t_2} \\ & \quad + \int_{B_R \times (t_1, t_2)} \left(-\|\sigma\|_{L^\infty}((\partial_t U)^2 + |\nabla U|^2) + \sigma(\partial_t U)^2 \right) e^{\|\sigma\|_{L^\infty}t} dx dt \\ & \quad + \frac{1}{2} \int_{\partial B_R \times (t_1, t_2)} ((\partial_\nu U)^2 + (\partial_t U)^2) e^{2\|\sigma\|_{L^\infty}t} ds(x) dt \\ & \leq E_0(t_2)e^{2\|\sigma\|_{L^\infty}t_2} + \frac{1}{2} \int_{\partial B_R \times (t_1, t_2)} ((\partial_\nu U)^2 + (\partial_t U)^2) e^{2\|\sigma\|_{L^\infty}t} ds(x) dt, \end{aligned}$$

and therefore

$$(A.6) \quad E_0(t_1) \leq e^{2\|\sigma\|_{L^\infty}(t_2-t_1)} (E_0(t_2) + F^2).$$

Now we replace E_0 by E and consider $t < s < t_2$ first. We have

$$\begin{aligned} \int_{B_R} U^2(x, s) dx &= \int_{B_R} U^2(x, t_2) dx - 2 \int_{B_R \times (s, t_2)} U(x, t) \partial_t U(x, t) dx dt \\ &\leq \int_{B_R} U^2(x, t_2) dx + \frac{1}{2(t_2 - t)} \int_{B_R \times (s, t_2)} U^2(x, t) dx dt \\ &\quad + 2(t_2 - t) \int_{B_R \times (s, t_2)} (\partial_t U(x, t))^2 dx dt. \end{aligned}$$

Integrating over (t, t_2) and interchanging the time integration order, we obtain

$$\begin{aligned} & \int_{B_R \times (t, t_2)} U^2(x, s) dx ds \\ & \leq (t_2 - t) \int_{B_R} U^2(x, t_2) dx + \frac{1}{2(t_2 - t)} \int_{B_R \times (t, t_2)} (s - t) U^2(x, s) dx ds \\ & \quad + 2(t_2 - t) \int_{B_R \times (t, t_2)} (s - t) (\partial_t U)^2(x, s) dx ds \\ & \leq (t_2 - t) \int_{B_R} U^2(x, t_2) dx + \frac{1}{2} \int_{B_R \times (t, t_2)} U^2(x, s) dx ds \\ & \quad + 2(t_2 - t)^2 \int_{B_R \times (t, t_2)} (\partial_t U(x, s))^2 dx ds. \end{aligned}$$

As a consequence we have

$$\begin{aligned} & \int_{B_R \times (t, t_2)} U^2(x, s) dx ds \\ & \leq 2(t_2 - t) \int_{B_R} U^2(x, t_2) dx + 4(t_2 - t)^2 \int_{B_R \times (t, t_2)} (\partial_t U(x, s))^2 dx ds \end{aligned}$$

and finally

$$(A.7) \quad \int_{B_R} U^2(x, t) dx \leq 2 \int_{B_R} U^2(x, t_2) dx + 4(t_2 - t) \int_{B_R \times (t, t_2)} (\partial_t U(x, s))^2 dx ds.$$

Similarly, when $t_1 < t < t_2$ we obtain

$$(A.8) \quad \int_{B_R} U^2(x, t) dx \leq 2 \int_{B_R} U^2(x, t_1) dx + 4(t - t_1) \int_{B_R \times (t_1, t)} (\partial_t U(x, s))^2 dx ds.$$

Combining (A.6) and (A.7), we conclude that

$$(A.9) \quad E(t) \leq 2e^{2\|\sigma\|_{L^\infty}(t_2-t_1)}(E(t_2) + F^2) + 4(t_2 - t_1) \int_{(t, t_2)} E(s) ds, \quad t_1 < t < t_2.$$

We recall that the inequality $E(t) \leq F_1 + A \int_t^{t_2} E(s) ds$, $t_1 < t < t_2$, implies that $E(t) \leq F_1 e^{A(t_2-t)}$. Using this result, we derive from (A.9) the bound (A.4).

Similarly, from (A.5) and (A.8) we obtain the bound (A.3). $\square$

After these preparations we return to the proof of Theorem A.1.

Proof of Theorem A.1. Let $\varphi(x, t) = |x - a|^2 - \rho^2(t - \frac{T}{2})^2$, where $\text{dist}(a, B_R) = 1$, $\rho = \frac{1}{2}$. We use the known Carleman-type estimate in [16] by denoting $Q = B_R \times (0, T)$:

$$(A.10) \quad \begin{aligned} & C^{-1} \sum_{|\alpha| \leq 1} \tau^{3-2|\alpha|} \int_Q |\partial^\alpha U|^2 e^{2\tau\varphi} dx dt \\ & \leq \int_Q ((\partial_t^2 - \Delta)U)^2 e^{2\tau\varphi} dx dt \\ & \quad + \int_{\partial Q} \tau((\partial_t U)^2 + |\nabla U|^2 + \tau^2 U^2) e^{2\tau\varphi} ds(x) dt. \end{aligned}$$

Obviously, $1 - \rho^2 \varepsilon_1^2 \leq \varphi$ on $B_R \times \{|t - \frac{T}{2}| < \varepsilon_1\}$ for some positive $\varepsilon_1 < 1$, so it holds that

$$(A.11) \quad \begin{aligned} & \sum_{|\alpha| \leq 1} \tau^{3-2|\alpha|} \int_Q |\partial^\alpha U|^2 e^{2\tau\varphi} dx dt \\ & \geq \sum_{|\alpha| \leq 1} \tau^{3-2|\alpha|} \int_{B_R \times (\frac{T}{2} - \varepsilon_1, \frac{T}{2} + \varepsilon_1)} |\partial^\alpha U|^2 e^{2\tau(1-\rho^2\varepsilon_1^2)} dx dt \\ & \geq \tau e^{2\tau(1-\rho^2\varepsilon_1^2)} \int_{(\frac{T}{2} - \varepsilon_1, \frac{T}{2} + \varepsilon_1)} E(t) dt \\ & \geq \tau e^{2\tau(1-\rho^2\varepsilon_1^2)\varepsilon_1} \left(\frac{1}{2} e^{-(2\|\sigma\|_{L^\infty} + 4T)T} E(0) - F^2 \right) \end{aligned}$$

due to (A.4).

Moreover, since $\varphi \leq (2R+1)^2 - \rho^2 T^2/4$ on $B_R \times \{0, T\}$, we have

$$(A.12) \quad \begin{aligned} & \int_{B_R \times \{0, T\}} (|\partial_t U|^2 + |\nabla U|^2 + \tau^2 U^2) e^{2\tau\varphi} dx dt \\ & \leq \tau^2 e^{2\tau((2R+1)^2 - \rho^2 T^2/4)} (E(0) + E(T)) \\ & \leq \tau^2 e^{2\tau((2R+1)^2 - \rho^2 T^2/4)} \left((e^{4T^2} + 1)E(0) + e^{4T^2} F^2 \right) \end{aligned}$$

due to (A.3). From (A.10) and (A.11) we obtain

(A.13)

$$\begin{aligned} & \tau e^{2\tau(1-\rho^2\varepsilon_1^2)} \varepsilon_1 e^{-(2\|\sigma\|_{L^\infty}+4T)T} E(0) + \sum_{|\alpha|\leq 1} \tau^{3-2|\alpha|} \int_Q |\partial^\alpha U|^2 e^{2\tau\varphi} \\ & \leq C \left(\|\sigma\|_{L^\infty}^2 \int_Q |\partial_t U|^2 e^{2\tau\varphi} dx dt + \int_{\partial B_R \times (0,T)} \tau((\partial_t U)^2 + |\nabla U|^2 + \tau^2 U^2) e^{2\tau\varphi} ds(x) dt \right. \\ & \quad + (\tau e^{2\tau(1-\rho^2\varepsilon_1^2)} + \tau^3 e^{2\tau((2R+1)^2-\rho^2 T^2/4)} e^{4T^2}) F^2 \\ & \quad \left. + \tau^3 e^{2\tau((2R+1)^2-\rho^2 T^2/4)} e^{4T^2} E(0) \right). \end{aligned}$$

Choosing $\tau > \frac{C\|\sigma\|_{L^\infty}^2}{2}$, we absorb the first integral on the right-hand side by the left-hand side and eliminate this integral. We further choose $T^2 = 4\frac{(2R+1)^2}{\rho^2} + 4\varepsilon_1^2$; then $\tau = (2\|\sigma\|_{L^\infty} + 8T)T + \ln(2(\varepsilon_1)^{-1}C) + C\|\sigma\|_{L^\infty}^2$. As a consequence,

$$\begin{aligned} \tau^2 e^{2\tau((2R+1)^2-\rho^2 T^2/4-1+\rho^2\varepsilon_1^2)+(2\|\sigma\|_{L^\infty}+8T)T} &= \tau^2 e^{-2\tau+(2\|\sigma\|_{L^\infty}+8T)T} \\ &\leq 2e^{-\tau+(2\|\sigma\|_{L^\infty}+8T)T} \leq \frac{\varepsilon_1}{2C} \end{aligned}$$

because $\tau^2 e^{-\tau} \leq 2$. In addition,

$$\begin{aligned} & \tau^2 e^{2\tau((2R+1)^2-1+\rho^2\varepsilon_1^2)+(2\|\sigma\|_{L^\infty}+4T)T} \\ & \leq \tau^2 e^{2((2\|\sigma\|_{L^\infty}+8T)T+C\|\sigma\|_{L^\infty}^2)+(2R+1)^2+(2\|\sigma\|_{L^\infty}+4T)T} \\ & \leq C e^{C\|\sigma\|_{L^\infty}^2} \end{aligned}$$

because $T < 5(2R+1)$. Using the above inequality and the inequality $\varphi < (2R+1)^2$ on $B_R \times (0, T)$ and dividing both sides in (A.13) by the factor of $E(0)$ on the left-hand side, we obtain

$$E(0) \leq C e^{C\|\sigma\|_{L^\infty}^2} F^2.$$

Since f is supported in B_R , it holds that $\|U\|_{(0)}(\partial B_R \times (0, T)) \leq C\|\partial_t U\|_{(0)}(\partial B_R \times (0, T))$, and we complete the proof of Theorem A.1. $\square$

REFERENCES

- [1] S. ACOSTA, S. CHOW, J. TAYLOR, AND V. VILLAMIZAR, *On the multi-frequency inverse source problem in heterogeneous media*, Inverse Problems, 28 (2012), 075013, <https://doi.org/10.1088/0266-5611/28/7/075013>.
- [2] R. ALBANESE AND P. MONK, *The inverse source problem for Maxwell's equations*, Inverse Problems, 22 (2006), pp. 1023–1035, <https://doi.org/10.1088/0266-5611/22/3/018>.
- [3] G. BAO, J. LIN, AND F. TRIKI, *A multi-frequency inverse source problem*, J. Differential Equations, 249 (2010), pp. 3443–3465, <https://doi.org/10.1016/j.jde.2010.08.013>.
- [4] G. BAO, P. LI, J. LIN, AND F. TRIKI, *Inverse scattering problems with multi-frequencies*, Inverse Problems, 31 (2015), 093001, <https://doi.org/10.1088/0266-5611/31/9/093001>.
- [5] G. BAO, P. LI, AND Y. ZHAO, *Stability for the inverse source problems in elastic and electromagnetic waves*, J. Math. Pures Appl., 134 (2020), pp. 122–178, <https://doi.org/10.1016/j.matpur.2019.06.006>.
- [6] C. CARLOS, N. MARTINS, AND N. ROBERTY, *Full identification of acoustic sources with multiple frequencies and boundary measurements*, Inverse Probl. Imaging, 3 (2009), pp. 275–294, <https://doi.org/10.3934/ipi.2009.3.275>.

- [7] D. COLTON AND R. KRESS, *Inverse Acoustic and Electromagnetic Scattering Theory*, Appl. Math. Sci. 93, Springer-Verlag, Berlin, 1998.
- [8] J.-C. NEDELEC, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Appl. Math. Sci. 144, Springer-Verlag, New York, 2001.
- [9] Z. CHEN AND X. LIU, *An adaptive perfectly matched layer technique for time-harmonic scattering problems*, SIAM J. Numer. Anal., 43 (2005), pp. 645–671, <https://doi.org/10.1137/040610337>.
- [10] D. COLTON AND R. KRESS, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd ed., Springer, New York, 2013.
- [11] J. CHENG, V. ISAKOV, AND S. LU, *Increasing stability in the inverse source problem with many frequencies*, J. Differential Equations, 260 (2016), pp. 4786–4804, <https://doi.org/10.1016/j.jde.2015.11.030>.
- [12] G. DASSIOS AND F. KARIOYOU, *Magnetoencephalography in ellipsoidal geometry*, J. Math. Phys., 44 (2003), pp. 220–241, <https://doi.org/10.1063/1.1522135>.
- [13] A. J. DEVANEY, E. A. MARENGO, AND M. LI, *Inverse source problem in nonhomogeneous background media*, SIAM J. Appl. Math., 67 (2007), pp. 1353–1378, <https://doi.org/10.1137/060658618>.
- [14] S. DYATLOV AND M. ZWORSKI, *Mathematical Theory of Scattering Resonances*, Grad. Stud. Math. 200, AMS, Providence, RI, 2019.
- [15] M. ELLER AND N. VALDIVIA, *Acoustic source identification using multiple frequency information*, Inverse Problems, 25 (2009), 115005, <https://doi.org/10.1088/0266-5611/25/11/115005>.
- [16] V. ISAKOV, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, New York, 2006.
- [17] V. ISAKOV AND S. LU, *Increasing stability in the inverse source problem with attenuation and many frequencies*, SIAM J. Appl. Math., 78 (2018), pp. 1–18, <https://doi.org/10.1137/17M1112704>.
- [18] S. KAWASHIMA, M. NAKAO, AND K. ONO, *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Jpn., 47 (1995), pp. 617–653.
- [19] S. KUSIAK AND J. SYLVESTER, *The scattering support*, Comm. Pure Appl. Math., 56 (2003), pp. 1525–1548, <https://doi.org/10.1002/cpa.3038>.
- [20] S. KUSIAK AND J. SYLVESTER, *The convex scattering support in a background medium*, SIAM J. Math. Anal., 36 (2005), pp. 1142–1158, <https://doi.org/10.1137/S0036141003433577>.
- [21] P. LI, J. ZHAI, AND Y. ZHAO, *Stability for the acoustic inverse source problem in inhomogeneous media*, SIAM J. Appl. Math., 80 (2020), pp. 2547–2559, <https://doi.org/10.1137/20M1334267>.
- [22] I. RYO, *Fast decay of solutions for linear wave equations with dissipation localized near infinity in an exterior domain*, J. Differential Equations, 188 (2003), pp. 390–405, [https://doi.org/10.1016/S0022-0396\(02\)00101-8](https://doi.org/10.1016/S0022-0396(02)00101-8).
- [23] M. ZWORSKI, *Semiclassical Analysis*, Grad. Stud. Math. 138, AMS, Providence, RI, 2012.