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Inverse random source problems for semilinear stochastic biharmonic wave equations

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Abstract

This paper concerns the direct and inverse source scattering problems for the stochastic biharmonic wave equation with a Lipschitz-type nonlinearity. The driven source is assumed to be a white noise. Given the rough random source, the direct problem is shown to be well-posed in the sense of distributions. For the inverse problem, it is shown that the strength of the random source can be uniquely determined by the expectation of the high frequency limit of the correlated far-field pattern. Moreover, we extend the results to the more general case where the driven source is a generalized Gaussian random function whose covariance operator is a classical pseudodifferential operator. For this case, we demonstrate by ergodicity that the strength of the random source can be uniquely determined by a single realization of the far-field pattern almost surely. This paper presents the first uniqueness result on inverse source scattering problems for nonlinear wave equations.

Keywords: inverse source problem, stochastic wave equation, Lipschitz-type nonlinearity, Gaussian random field, uniqueness

Mathematics subject classification: 35R60, 35R30, 78A46, 60H15

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1. Introduction

Inverse source problem aims at determining the unknown source from measurements of the radiated field away from its support. It is an important research subject in inverse problems which arises in many scientific and industrial areas such as antenna design and synthesis, biomedical imaging, and elasticity. There have been many theoretical and numerical results on inverse source problems for various wave equations. We refer to [2, 12] and references therein for deterministic inverse source problems, and the recent works [1, 14, 15, 17] for the stochastic counterparts.

In this paper, we consider the semilinear stochastic biharmonic wave equation driven by an additive white noise:

$$\Delta^2 u(x) - k^4 u(x) + V(x)f(u(x)) = g(x), \quad x \in \mathbb{R}^2, \quad (1.1)$$

where $g = \sigma(x)\dot{W}(x)$ with $\dot{W}$ being a white noise and W being the Brownian motion defined in the complete probability space $(\Omega, \mathcal{F}, P)$ (see [1, appendix A]), and $V \in C_0^\infty(\mathbb{R}^2)$. $\sigma \in C_0^\infty(\mathbb{R}^2)$ is called the strength of the random source satisfying $\sigma \geq 0$ and f is a Lipschitz function with Lipschitz constant L and $f(0) = 0$. We assume that the supports of σ and V are compactly supported in a disk D centered at the origin with radius R . The scattered field u and its Laplacian Δu are required to satisfy the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{\frac{1}{2}} (\partial_r u - iku) = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{1}{2}} (\partial_r (\Delta u) - ik(\Delta u)) = 0 \quad r = |x|. \quad (1.2)$$

We refer to [16] for the radiation condition which is used to guarantee the uniqueness of the direct scattering problem.

Biharmonic wave equation and its nonlinear generalizations arise in many scientific and industrial areas such as thin plate elasticity and quantum mechanics [3, 4, 13, 19, 22]. For instance, in thin plate elasticity, they have found applications in the control of destructive surface waves and the construction of ultrabroadband elastic cloaking devices [10, 11, 25]. The biharmonic wave equation with a Lipschitz-type nonlinearity also serves as a mathematical model for nonlinearly supported bending beams, which is used to investigate the nonlinear oscillations in suspension bridges [5, 20].

Driven by the important applications, scattering problems for biharmonic wave equations have attracted much attention. In particular, in situations involving unpredictable environments, incomplete knowledge of the system, and random measurement noise, it is more appropriate to assume that the driven source is created by random process [9]. As a consequence, the radiated field is also random. We refer to [14, 16, 17] for recent advancements on scattering problems for stochastic biharmonic wave equations. However, all the aforementioned works consider linear biharmonic equations, while little is known for nonlinear cases.

In this work, we establish the well-posedness of the direct scattering problem (1.1)–(1.2). Since white noise in two dimensions belongs to the Sobolev space $H^{-1-\epsilon}$ which has a negative smoothness index, it is too rough to exist pointwisely and should be understood as a distribution. Hence, compared with the deterministic settings, the direct problem requires a more careful analysis. The analysis employs resolvent estimates for the resolvent of the biharmonic operator obtained by [24, lemma 3.3].

Based on the analysis of the direct problem, we also consider the inverse random source problem. It should be noticed that compared with the deterministic cases, it is now less meaningful to characterize the random source by a particular realization, while the mean and variance of the random source are of more interests which are used to quantify the uncertainties of the source. Therefore, in this work we study the inverse problem of determining the strength

σ from the random far-field pattern. To the best of our knowledge, inverse source scattering problem for nonlinear wave equations remains widely open so far. In fact, due to the nonlinearity of the equation, the inverse source problem becomes nonlinear and existing methods for linear equations are no longer applicable. We demonstrate that the strength can be determined by the expectation of the high frequency limit of the correlated far-field pattern.

As a further study, we consider the more general case where the random source has the form $\sqrt{\mu}(I - \Delta)^{-\frac{m}{4}}\dot{W}$. In fact, this random field can be described as a generalized Gaussian random field whose covariance operator is a classical pseudodifferential operator with principle symbol of the form $\mu(x)|\xi|^{-m}$ (see [6, example 2.2]). Recently, such kind of random fields have been extensively investigated for linear stochastic wave equations [6, 15–17]. With the aid of ergodicity, we demonstrate that the principle symbol can be uniquely determined by the far-field pattern for the currently investigated nonlinear wave equation from a single realization of the random source almost surely.

This paper is organized as follows. In section 2, we establish the well-posedness of the direct problem. Section 3 is devoted to the inverse random source problem, where the random source is either a white noise or a generalized Gaussian random field. Useful resolvent estimates of the biharmonic operator for rough fields are presented in appendix A, and we introduce the generalized Gaussian random field and its relevant properties in appendix B.

2. Well-posedness of the direct scattering problem

In this section, we establish the well-posedness of the direct scattering problem for each realization of the rough random field $\sigma\dot{W}$ by constructing a convergent sequence using the Lippmann–Schwinger integral equation.

Throughout the paper, we use the notation $a \lesssim b$ to stand for $a \leq Cb$, where $C > 0$ is a generic constant whose specific value is not required but should be clear from the context.

In view of the decomposition formula

$$(\Delta^2 - k^4)^{-1} = \frac{1}{2k^2} \left((-\Delta - k^2)^{-1} - (-\Delta + k^2)^{-1} \right),$$

the Green function of the biharmonic equation has the form

$$G(x, y) = \frac{1}{2k^2} \left(\frac{i}{4} H_0^{(1)}(k|x - y|) - \frac{i}{4} H_0^{(1)}(ik|x - y|) \right).$$

Following the arguments in [15, theorem 3.2], it can be verified that a solution $u \in H_{\text{loc}}^2(\mathbb{R}^2)$ to the following Lippmann–Schwinger integral equation also satisfies the direct problem (1.1) and (1.2) in the sense of distribution:

$$u = \int_D G(x, y) g(y) dy - \int_D G(x, y) Vf(u(y)) dy := u_0 - \int_D G(x, y) Vf(u(y)) dy. \quad (2.1)$$

Therefore, in what follows, we show the existence of a solution to the above integral equation. To this end, motivated by [23], we introduce the sequence

$$u_{j+1} = u_0 - \int_D G(x, y) Vf(u_j) dy. \quad (2.2)$$

The strategy is to prove that the sequence u_j converges to a solution to (2.1).

Lemma 2.1. *There exists a constant c_0 depending on the Lipschitz constant L and the domain D such that for any $k > k_0$ with $k_0 \geq (c_0 \|V\|_{L^\infty})^{2/5}$, the following estimate holds*

$$\|u_j\|_{L^\infty(D)} \leq c_1 k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)}, \quad (2.3)$$

where $0 < \delta < \frac{1}{2}$. The constant c_1 only depends on ϵ, δ, D, L and k_0 .

Proof. We prove the lemma by induction. It is known that $g = \sigma \dot{W} \in H^{-1-\epsilon}(D)$ where $\epsilon > 0$ is any positive constant. Noting $H^s(D) \subset C(D)$ for $s > 1$ by Sobolev embedding theorem, it follows from the resolvent estimates in lemma A.1 that for each constant δ with $0 < \delta < 1/2$ it holds

$$\|u_0\|_{L^\infty(D)} \leq C_1 k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)},$$

where C_1 is a constant, which only depends on ϵ, δ and D .

Noting that f is Lipschitz, we have

$$\begin{aligned} \int_D |G(x, y) V(y) f(v)(y)| dy &\leq \int_D L |G(x, y) V(y) v(y)| dy \\ &\leq L \left(\int_D |G(x, y)| dy \right) \|V\|_{L^\infty(D)} \|v\|_{L^\infty(D)} \\ &\leq C_2 k^{-5/2} \|V\|_{L^\infty(D)} \|v\|_{L^\infty(D)}, \end{aligned} \quad (2.4)$$

where the constant C_2 only depends on L and D . Here we have used the following estimate for the Hankel function (see [8])

$$|H_0^{(1)}(k|x-y|)| \lesssim \frac{1}{\sqrt{k|x-y|}}, \quad k > 0.$$

Based on the above analysis, if k_0 is a positive constant satisfying $k_0 > (C_2 \|V\|_{L^\infty(D)})^{2/5}$, then we have for $k > k_0$

$$\begin{aligned} \|u_j\|_{L^\infty(D)} &\leq C_1 k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)} + C_2 k^{-5/2} \|V\|_{L^\infty(D)} \|u_j\|_{L^\infty(D)} \\ &\leq \left(1 + C_2 k^{-5/2} \|V\|_{L^\infty(D)} + \left(C_2 k^{-5/2} \|V\|_{L^\infty(D)} \right)^2 + \dots \right. \\ &\quad \left. + \left(C_2 k^{-5/2} \|V\|_{L^\infty(D)} \right)^j \right) C_1 k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)} \\ &\leq \frac{C_1}{1 - C_2 k^{-5/2} \|V\|_{L^\infty(D)}} k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)} \\ &\leq \frac{C_1}{1 - C_2 k_0^{-5/2} \|V\|_{L^\infty(D)}} k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)}. \end{aligned}$$

Therefore, (2.3) holds with $c_0 = C_2$ and

$$c_1 := \frac{C_1}{1 - C_2 k_0^{-5/2} \|V\|_{L^\infty(D)}}.$$

The proof is complete. $\square$

Lemma 2.2. *The sequence $\{u_j\}_{j=1}^\infty$ converges to a bounded solution u of the Lippmann–Schwinger equation in $L^\infty(D)$ for $k \geq k_0$ with the estimate*

$$\|u\|_{L^\infty(D)} \leq c_1 k^{-\delta} \|g\|_{H^{-1-\epsilon}(D)},$$

where the constants c_1 and k_0 are specified in lemma 2.1.

Proof. Similarly as (2.4), we have

$$\begin{aligned} |u_{j+1}(x) - u_j(x)| &\leq \int_D |G(x, y) V(y) (f(u_j)(y) - f(u_{j-1})(y))| dy \\ &\leq \int_D L |G(x, y) V(y) (u_j(y) - u_{j-1}(y))| dy \\ &\leq L \|V\|_{L^\infty(D)} \left(\int_D |G(x, y)| dy \right) \|u_j - u_{j-1}\|_{L^\infty(D)} \\ &\leq C_2 \|V\|_{L^\infty(D)} k^{-5/2} \|u_j - u_{j-1}\|_{L^\infty(D)}. \end{aligned}$$

Then we obtain

$$\|u_{j+1} - u_j\|_{L^\infty(D)} \leq C_2 \|V\|_{L^\infty(D)} k^{-5/2} \|u_j - u_{j-1}\|_{L^\infty(D)}. \quad (2.5)$$

Now we show that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^\infty(D)$. For $m, n \in \mathbb{N}$ with $m > n$, we have from (2.5)

$$\begin{aligned} \|u_m - u_n\|_{L^\infty(D)} &\leq \sum_{i=n}^{m-1} \|u_{i+1} - u_i\|_{L^\infty(D)} \\ &\leq \left(\left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^{m-1} + \left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^{m-2} + \dots \right. \\ &\quad \left. + \left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^{n-1} \right) \|u_1 - u_0\|_{L^\infty(D)} \\ &\leq \left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^m + \left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^{m-1} + \dots \\ &\quad + \left(C_2 \|V\|_{L^\infty(D)} k^{-5/2} \right)^n, \end{aligned}$$

which implies that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^\infty(D)$ provided $k \geq k_0 > (C_2 \|V\|_{L^\infty(D)})^{2/5}$. Assume that the limit of the sequence $\{u_j\}_{j=1}^\infty$ is u . Letting $j \rightarrow \infty$ in (2.2) yields the integral equation (2.1), and the desirable estimate follows from (2.3). The proof is complete. $\square$

The following result is a corollary of lemma 2.2, which is useful in the subsequent study of the inverse problem.

Lemma 2.3. *For $k \geq k_0$ it holds*

$$\mathbb{E} \left[\|u\|_{L^\infty(D)}^2 \right] \leq c_1 c_2 k^{-2\delta}, \quad (2.6)$$

where the constants c_1, k_0 are specified in lemma 2.1 and c_2 is a constant dependent on σ .

Proof. Using [26, lemma 10.3.7], we obtain

$$\mathbb{E} \left[\|g\|_{H^{-1-\epsilon}(D)}^2 \right] \leq c_2 \mathbb{E} \left[\|\dot{W}_x\|_{H^{-1-\epsilon}(D)}^2 \right] < \infty,$$

where the constant c_2 depends on σ . Moreover, from lemma 2.2 we have

$$\mathbb{E} \left[\|u\|_{L^\infty(D)}^2 \right] \leq c_1 c_2 k^{-2\delta} \mathbb{E} \left[\|g\|_{H^{-1-\epsilon}(D)}^2 \right],$$

which yields (2.6). The proof is complete. $\square$

The following theorem concerns the well-posedness of the direct scattering problem.

Theorem 2.4. *For $k \geq k_0$, the direct scattering problem admits a unique solution in the sense of distribution almost surely such that $u \in H_{\text{loc}}^{3-\epsilon}(\mathbb{R}^2)$ for any $\epsilon > 0$ which satisfies the integral equation*

$$u = \int_D G(x, y) \sigma(y) dW(y) - \int_D G(x, y) Vf(u) dy.$$

Proof. From lemma 2.2, there exists a solution $u \in L^\infty(D)$ to the integral equation (2.1). Since $V \in C_0^\infty(\mathbb{R}^2)$ and f is Lipschitz which yields $f(u) \in L^\infty(D)$, from [7, appendix] we obtain $\int_D G(x, y) Vf(u(y)) dy \in H_{\text{loc}}^4(\mathbb{R}^2)$. On the other hand, as $g \in H^{-1-\epsilon}(\mathbb{R}^2)$ with compact support in D , we have from lemma A.1 that $\int_D G(x, y) g(y) dy \in H_{\text{loc}}^{3-\epsilon}(\mathbb{R}^2)$. As a consequence, noting that D is a disk of radius R where R could be arbitrarily large, we derive $u \in H_{\text{loc}}^{3-\epsilon}(\mathbb{R}^2)$. Moreover, following the arguments in [15, theorem 3.2] we have that u is a solution to the equation (1.1) in the sense of distribution.

Now we prove the uniqueness. Assume that u_1 and u_2 are two solutions to the direct problem and let $u = u_1 - u_2$. Then u satisfies the equation $\Delta^2 u - k^4 u = 0$ in $\mathbb{R}^2 \setminus D$. According to the proof of [16, theorem 4.1], it can be shown $u = 0$ in $\mathbb{R}^2 \setminus D$, which yields $u \equiv 0$ in $\mathbb{R}^2$ by the unique continuation principle of deterministic nonlinear elliptic equations in [21]. $\square$

Remark 2.5. Theorem 2.4 shows that the direct problem is well-posed at high wavenumbers. However, given any $k > 0$, it can be seen from the proof of lemma 2.2 that theorem 2.4 still holds if the norm $\|V\|_{L^\infty}$ is sufficiently small.

3. Inverse random source problem

In this section, based on the analysis of the direct problem, we investigate the inverse random source by using the random far-field pattern. We also consider a more general class of random sources which is called the generalized Gaussian random field, whose covariance is a pseudodifferential operator.

3.1. White noise

In this section, we consider the inverse random source for the nonlinear biharmonic wave equation. The far-field pattern associated with the scattering problem (1.1) and (1.2) is

$$u^\infty(\hat{x}, k) = k^{-5/2} \int_D e^{-ik\hat{x} \cdot y} \sigma dW_y - k^{-5/2} \int_D e^{-ik\hat{x} \cdot y} Vf(u) dy, \quad (3.1)$$

where $\hat{x} = \frac{x}{|x|} \in \mathbb{S}$ is the observation direction. The inverse problem concerns determining the variance σ^2 from the far-field pattern of the random far-field pattern.

The following theorem on uniqueness of the inverse random source problem is the main result in this section.

Theorem 3.1. *For all $\hat{x} \in \mathbb{S}$ and $\tau \geq 0$, it holds almost surely that*

$$\lim_{k \rightarrow \infty} k^5 \mathbb{E} \left[u^\infty(\hat{x}, k + \tau) \overline{u^\infty(\hat{x}, k)} \right] = \widehat{\sigma^2}(\tau \hat{x}) \quad (3.2)$$

Moreover, σ^2 can be uniquely determined by (3.2) with $(\tau, \hat{x}) \in E$ and E being any open domain in $\mathbb{R}^+ \times \mathbb{S}$.

Proof. Denote

$$\begin{aligned} u^\infty(\hat{x}, k) &= k^{-5/2} \int_D e^{-ik\hat{x} \cdot y} \sigma dW_y - k^{-5/2} \int_D e^{-ik\hat{x} \cdot y} V f(u) dy \\ &=: I_1(\hat{x}, k) - I_2(\hat{x}, k). \end{aligned} \quad (3.3)$$

By taking the expectation of the correlation of the far-field pattern $u^\infty(\hat{x}, k)$ we obtain

$$\begin{aligned} \mathbb{E} \left[u^\infty(\hat{x}, k + \tau) \overline{u^\infty(\hat{x}, k)} \right] &= \mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_1(\hat{x}, k)} \right] - \mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right] \\ &\quad - \mathbb{E} \left[I_2(\hat{x}, k + \tau) \overline{I_1(\hat{x}, k)} \right] + \mathbb{E} \left[I_2(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right]. \end{aligned}$$

We estimate the four terms in the above sum. First, from Ito isometry we have

$$\mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_1(\hat{x}, k)} \right] = k^{-5} \int_D \sigma^2 e^{-i\tau \hat{x} \cdot y} dy. \quad (3.4)$$

For the second term, from the Hölder inequality it holds that

$$\left| \mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right] \right| \leq \left(\mathbb{E} \left[|I_1(\hat{x}, k + \tau)|^2 \right] \right)^{1/2} \left(\mathbb{E} \left[|I_2(\hat{x}, k)|^2 \right] \right)^{1/2}.$$

Then using Ito isometry again we have

$$\mathbb{E} \left[|I_1(\hat{x}, k + \tau)|^2 \right] = \mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_1(\hat{x}, k + \tau)} \right] = k^{-5} \int_D \sigma^2 dy$$

and from the resolvent estimate (2.6)

$$\mathbb{E} \left[|I_2(\hat{x}, k)|^2 \right] \lesssim k^{-5} \mathbb{E} \left[\|u(\omega, k)\|_{L^\infty(D)}^2 \right] \lesssim k^{-5-2\delta},$$

which yields

$$\left| \mathbb{E} \left[I_1(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right] \right| \lesssim k^{-5-\delta}. \quad (3.5)$$

In a similar way, we also have

$$\left| \mathbb{E} \left[I_2(\hat{x}, k + \tau) \overline{I_1(\hat{x}, k)} \right] \right| \lesssim k^{-5-\delta}. \quad (3.6)$$

For the fourth term, since f is Lipschitz, it holds

$$\begin{aligned} \left| \mathbb{E} \left[I_2(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right] \right| &= k^{-5} \left| \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x} \cdot y} V f(u) dy \int_D e^{ik\hat{x} \cdot y} \overline{V f(u)} dy \right] \right| \\ &\leq k^{-5} \mathbb{E} \left[\|u(\omega, k)\|_{L^\infty(D)} \|u(\omega, k + \tau)\|_{L^\infty(D)} \right], \end{aligned}$$

then we obtain from Hölder inequality and (2.6)

$$\begin{aligned} &\mathbb{E} \left[\|u(\omega, k)\|_{L^\infty(D)} \|u(\omega, k + \tau)\|_{L^\infty(D)} \right] \\ &\leq \left(E \left[\|u(\omega, k)\|_{L^\infty(D)}^2 \right] \right)^{1/2} \left(E \left[\|u(\omega, k + \tau)\|_{L^\infty(D)}^2 \right] \right)^{1/2} \leq C k^{-2\delta}, \end{aligned}$$

which gives

$$\left| \mathbb{E} \left[I_2(\hat{x}, k + \tau) \overline{I_2(\hat{x}, k)} \right] \right| \leq C k^{-5-2\delta}. \quad (3.7)$$

Combining (3.4)–(3.7), we arrive at for each $\tau > 0$

$$k^{-5} \int_D \sigma^2 e^{-i\tau\hat{x} \cdot y} dy = k^{-5} \mathbb{E} \left[u^\infty(\hat{x}, k + \tau) \overline{u^\infty(\hat{x}, k)} \right] + O(k^{-5-\delta}).$$

We obtain (3.2) by multiplying the above identity by k^5 and letting $k \rightarrow \infty$.

Since $\widehat{\sigma^2}$ is analytic, it can be uniquely determined by $\widehat{\sigma^2}(\tau\hat{x})$ for $(\tau, \hat{x}) \in E$ with E being any open domain in $\mathbb{R}^+ \times \mathbb{S}$. The proof is complete. $\square$

3.2. Generalized Gaussian random field

In this section, we consider a more general random source given by

$$g = \sqrt{\mu} (I - \Delta)^{-\frac{m}{4}} \dot{W},$$

where $\mu \in C_0^\infty(\mathbb{R}^2)$ and $\mu \geq 0$. The fractional differential operator $(I - \Delta)^{\frac{m}{4}}$ is defined by

$$(I - \Delta)^{-\frac{m}{4}} u := \mathcal{F}^{-1} \left[(1 + |\xi|^2)^{-\frac{m}{4}} \mathcal{F}(u) \right]. \quad (3.8)$$

It was shown in [6, example 2.2] that the random process defined by (3.8) is a generalized Gaussian random field of order $-m$, whose covariance operator is a classical pseudodifferential operator with principle symbol $\mu(x)|\xi|^{-m}$. For details on the generalized Gaussian random field we refer to appendix B. Moreover, we have the estimate

$$\mathbb{E} \left[\|g\|_{H^{-1+\frac{m}{2}-\epsilon}}^2 \right] \lesssim \mathbb{E} \left[\left\| (I - \Delta)^{-\frac{m}{4}} \dot{W} \right\|_{H^{-1+\frac{m}{2}-\epsilon}}^2 \right] = \mathbb{E} \left[\|\dot{W}\|_{H^{-1-\epsilon}}^2 \right] < \infty.$$

By following the arguments in the proof of theorem 2.4, we obtain the well-posedness of the direct scattering problem with a generalized Gaussian random source field.

Theorem 3.2. Suppose that $g = \sqrt{\mu} (I - \Delta)^{-\frac{m}{4}} \dot{W}$ with $m \geq 0$. For $k \geq k_0$, the direct scattering problem (1.1) and (1.2) admits a unique solution in the sense of distribution almost surely such that $u \in H_{\text{loc}}^{3-\epsilon+m/2}(\mathbb{R}^2)$ for any $\epsilon > 0$ which satisfies the integral equation

$$u = \int_D G(x, y) g(y) dy - \int_D G(x, y) V(y) f(u(y)) dy.$$

Moreover, it holds

$$\left(\mathbb{E}\|u\|_{L^\infty(D)}^2\right)^{\frac{1}{2}} \lesssim \begin{cases} k^{-\delta-m/2}, & m \leq 4, \\ k^{-5/2}, & m > 4, \end{cases}$$

where δ is an arbitrary positive constant in $(0, 1/2)$.

Remark 3.3. It can be seen from lemma A.1 that the decay of the term u_0 given in (2.1) with respect to k is at most $1/k^{5/2}$. When $m > 4$, we have $g \in H^s$ with $s > 1$ which means $g \in L^\infty(D)$. Therefore, from lemma A.1 we obtain $u_0 \in H_{\text{loc}}^s(\mathbb{R}^2) \subset L^\infty(D)$ with $\|u_0\|_{L^\infty(D)} \lesssim k^{-5/2}\|g\|_{H^{-5/2}(D)}$. Then applying similar arguments as lemma 2.1–2.3 gives $(\mathbb{E}\|u\|_{L^\infty(D)}^2)^{1/2} \lesssim k^{-5/2}\|g\|_{H^{-5/2}(D)}$. On the other hand, if $m \leq 4$, then g might not be in $L^\infty(D)$. In this case, we need to improve the regularity of u_0 such that $u_0 \in L^\infty(D)$ at the expense of a loss in the decay with respect to k , which results in $\|u_0\|_{L^\infty(D)} \lesssim k^{-\delta-m/2}\|g\|_{H^{-5/2}(D)}$. This accounts for the difference of the estimates of $\mathbb{E}\|u\|_{L^\infty(D)}^2$ in theorem 3.2 for $m > 4$ and $m \leq 4$.

The following lemma shows that the Fourier transform of the strength μ in the principle symbol can be determined by the expectation of the high frequency limit of the far-field pattern.

Lemma 3.4. Assume $g = \sqrt{\mu}(I - \Delta)^{-\frac{m}{4}}\dot{W}$ with $m \in [0, 5)$. For all $\hat{x} \in \mathbb{S}$ and $\tau \geq 0$, it holds

$$\lim_{k \rightarrow \infty} k^{m+5} \mathbb{E} \left[\overline{u^\infty(\hat{x}, k)} u^\infty(\hat{x}, k + \tau) \right] = \hat{\mu}(\tau \hat{x}). \quad (3.9)$$

Moreover, μ can be uniquely determined by (3.9) with $(\tau, \hat{x}) \in E$ and E being any open domain in $\mathbb{R}^+ \times \mathbb{S}$.

Proof. Since the random source is a generalized Gaussian random field, by taking correlation of the far-field pattern we obtain

$$\begin{aligned} \mathbb{E} \left[\overline{u^\infty(\hat{x}, k)} u^\infty(\hat{x}, k + \tau) \right] &= (k + \tau)^{-5/2} k^{-5/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i(k+\tau)\hat{x} \cdot y} e^{ik\hat{x} \cdot z} \mathbb{E} \left[g(y) \overline{g(z)} \right] dy dz \\ &\quad + (k + \tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x} \cdot y} g(y) dy \int_D e^{ik\hat{x} \cdot y} \overline{Vf(u)} dy \right] \\ &\quad + (k + \tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x} \cdot y} Vf(u) dy \int_D e^{ik\hat{x} \cdot y} g(y) dy \right] \\ &\quad + (k + \tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x} \cdot y} Vf(u) dy \int_D e^{ik\hat{x} \cdot y} \overline{Vf(u)} dy \right]. \end{aligned} \quad (3.10)$$

For the first term in the sum (3.10), we have

$$\begin{aligned} &(k + \tau)^{-5/2} k^{-5/2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i(k+\tau)\hat{x} \cdot y} e^{ik\hat{x} \cdot z} \mathbb{E} \left[g(y) \overline{g(z)} \right] dy dz \\ &= (k + \tau)^{-5/2} k^{-5/2} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} K_g(y, z) e^{-ik\hat{x} \cdot (y-z)} dz \right] e^{-i\tau\hat{x} \cdot y} dy \\ &= (k + \tau)^{-5/2} k^{-5/2} \left(\int_{\mathbb{R}^2} \mu(y) e^{-i\tau\hat{x} \cdot y} dy |k\hat{x}|^{-m} + O(k^{-m-1}) \right) \\ &= \left(\frac{k}{k + \tau} \right)^{5/2} k^{-5-m} \hat{\mu}(\tau \hat{x}) + O(k^{-6-m}). \end{aligned} \quad (3.11)$$

For the second term,

$$\begin{aligned}
 & \left| \mathbb{E} \left[\int_{\mathbb{R}^2} e^{-i(k+\tau)\hat{x}\cdot y} g(y) dy \int_{\mathbb{R}^2} e^{ik\hat{x}\cdot y} \overline{Vf(u)} dy \right] \right| \\
 & \leq \left(\mathbb{E} \left[\left| \int_{\mathbb{R}^2} e^{-i(k+\tau)\hat{x}\cdot y} g(y) dy \right|^2 \right] \right)^{1/2} \left(\mathbb{E} \left[\left| \int_{\mathbb{R}^2} e^{ik\hat{x}\cdot y} \overline{Vf(u)} dy \right|^2 \right] \right)^{1/2} \\
 & \leq \left(\mathbb{E} \left[\|u\|_{L^\infty(D)}^2 \right] \right)^{1/2} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i(k+\tau)\hat{x}\cdot(y-z)} \mathbb{E} [g(y) \overline{g(z)}] dy dz \right)^{1/2} \\
 & \lesssim \begin{cases} k^{-\delta-m}, & m \leq 4, \\ k^{-5/2-m/2}, & m > 4, \end{cases}
 \end{aligned}$$

with $\delta \in (0, 1/2)$, where theorem 3.2 and a similar estimate as in derivation of (3.11) are used in the last inequality. Then there holds

$$\begin{aligned}
 & (k+\tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x}\cdot y} g(y) dy \int_D e^{ik\hat{x}\cdot y} \overline{Vf(u)} dy \right] \\
 & \lesssim \begin{cases} k^{-\delta-m-5}, & m \leq 4, \\ k^{-(m+5)/2-5}, & m > 4. \end{cases} \quad (3.12)
 \end{aligned}$$

Similarly, it holds

$$\begin{aligned}
 & (k+\tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x}\cdot y} Vf(u) dy \int_D e^{ik\hat{x}\cdot y} g(y) dy \right] \\
 & \lesssim \begin{cases} k^{-\delta-m-5}, & m \leq 4, \\ k^{-(m+5)/2-5}, & m > 4. \end{cases} \quad (3.13)
 \end{aligned}$$

The last term can be estimated as follows

$$\begin{aligned}
 & (k+\tau)^{-5/2} k^{-5/2} \mathbb{E} \left[\int_D e^{-i(k+\tau)\hat{x}\cdot y} Vf(u) dy \int_D e^{ik\hat{x}\cdot y} \overline{Vf(u)} dy \right] \\
 & \lesssim \begin{cases} k^{-\delta-m-5}, & m \leq 4, \\ k^{-10}, & m > 4. \end{cases} \quad (3.14)
 \end{aligned}$$

Combining (3.11)–(3.14) and noting $m < 5$, multiplying both sides of (3.10) by k^{5+m} and letting $k \rightarrow \infty$ yields

$$\lim_{k \rightarrow \infty} k^{m+5} \mathbb{E} \left[\overline{u^\infty(\hat{x}, k)} u^\infty(\hat{x}, k+\tau) \right] = \hat{\mu}(\tau \hat{x}).$$

As a consequence, due to the analyticity of $\hat{\mu}$, it can be uniquely determined by $\hat{\mu}(\tau \hat{x})$ for $(\tau, \hat{x}) \in E$ with E being any open domain in $\mathbb{R}^+ \times \mathbb{S}$. $\square$

The above lemma shows that the strength μ can be uniquely determined from the expectation of the correlation far-field pattern, which requires the measurements at all sample paths of the random source. In the following, with the help of ergodicity, we show that μ can be recovered from far-field pattern at a single path almost surely.

The use of ergodicity is based on the following lemma [27, p 91]

Lemma 3.5. Let X_t with $t \geq 0$ be a real-valued stochastic process with continuous paths. In addition, the process X_t satisfies $\mathbb{E}X_t = 0$ and there exist non-negative constants c_0, c_1, c_2 , satisfying $0 \leq 2c_1 < c_2$ such that

$$|\mathbb{E}(X_s X_t)| \lesssim \frac{s^{c_1} + t^{c_1}}{1 + |s - t|^{c_2}}$$

for any $s, t \geq c_0$. Then we almost surely have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_T^{2T} X_t dt = 0.$$

Denote

$$u_0^\infty(\hat{x}, k) = k^{-5/2} \int_{\mathbb{R}^2} e^{-ik\hat{x} \cdot y} g(y) dy.$$

Then from the proof of theorem 3.4, there holds

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m+5} \mathbb{E} \left[u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} \right] dk = \hat{\mu}(\tau \hat{x}). \quad (3.15)$$

We next show that

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m+5} \left(u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} - \mathbb{E} \left[u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} \right] \right) dk = 0. \quad (3.16)$$

Denote $Y_k = u_0^\infty(\hat{x}, k)$. A direct calculation gives

$$u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} - \mathbb{E} \left[u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} \right] = \sum_{W_k \in \mathcal{A}} C(W_k) (W_k^2 - \mathbb{E}W_k^2), \quad (3.17)$$

where

$$\begin{aligned} \mathcal{A} := \{ & \Re Y_k, \Im Y_k, \Re Y_{k+\tau}, \Im Y_{k+\tau}, \Re Y_k - \Re Y_{k+\tau}, \\ & \Im Y_k - \Im Y_{k+\tau}, \Re Y_{k+\tau} + \Im Y_k, \Im Y_{k+\tau} - \Re Y_k \} \end{aligned}$$

and the constant $C(W_k) \in \{\frac{1+i}{2}, -\frac{1}{2}, -\frac{i}{2}\}$. Let $X_k := k^{m+5}(W_k^2 - \mathbb{E}W_k^2)$. According to [14, lemma 3.4], there holds

$$\mathbb{E}(X_{k_1} X_{k_2}) \lesssim (1 + |k_1 - k_2|)^{-N}$$

for all $N > 0$. Hence we can apply lemma 3.5 to obtain that

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} X_k ds = 0. \quad (3.18)$$

Thus, by combining (3.17) and (3.18) we have (3.16). In view of (3.15) and (3.16), we obtain

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m+5} u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} dk = \hat{\mu}(\tau \hat{x}).$$

Moreover, by using a similar argument as in the proof of theorem 3.4, we have

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m+5} \left(u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} - u_0^\infty(\hat{x}, k + \tau) \overline{u_0^\infty(\hat{x}, k)} \right) dk = 0.$$

In summary, the above analysis yields the following theorem on uniqueness of the inverse random source problem by a single realization.

Theorem 3.6. Suppose that $g = \sqrt{\mu}(I - \Delta)^{-\frac{m}{4}} \dot{W}$ with $m \in [0, 5)$. For all $\hat{x} \in \mathbb{S}$ and $\tau \geq 0$, it almost surely holds

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m+5} u^\infty(\hat{x}, k + \tau) \overline{u^\infty(\hat{x}, k)} dk = \widehat{\mu}(\tau \hat{x}). \quad (3.19)$$

Moreover, μ can be uniquely determined by (3.19) with $(\tau, \hat{x}) \in E$ and E being any open domain in $\mathbb{R}^+ \times \mathbb{S}$.

4. Conclusion

In this paper, we have investigated the direct and inverse random source problems for the nonlinear biharmonic wave equation. The source is assumed to be either a white noise or a Gaussian random function whose covariance operator is a classical pseudodifferential operator. For such rough sources, well-posedness is established in the sense of distribution. Uniqueness of the inverse random source problem is achieved by using the high frequency limit of the correlated far-field pattern from a single realization of the random source almost surely. We hope that the proposed method can be applied to study other stochastic nonlinear wave equations. For example, it is interesting to study random source scattering problems for stochastic acoustic and elastic wave equations. We hope to be able to report the progress on these problems in the future.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Resolvent estimates

In this section, we state resolvent estimates for the resolvent of the biharmonic operator given by [24, lemma 3.3], which plays an important role in the establishment of the well-posedness of the direct problem.

Denote the resolvent of the biharmonic operator by $R_0(\lambda) = (\Delta^2 - \lambda^4)^{-1}$ whose kernel is $G(x, y)$. Denote $\mathcal{S} := \mathbb{C} \setminus (-\infty, 0]i$ and

$$\alpha(\lambda) = \begin{cases} |\Re \lambda|, & \Im \lambda \leq 0, \\ |\lambda|, & \Im \lambda > 0. \end{cases}$$

The following lemma concerns the meromorphic continuation and estimates of $R_0(\lambda)$.

Lemma A.1. *The free resolvent $R_0(\lambda)$ is analytic for $\lambda \in \mathcal{S}$ with $\Im \lambda > 0, \Re \lambda > 0$ as a family of operators*

$$R_0(\lambda) : H^s(\mathbb{R}^2) \rightarrow H^t(\mathbb{R}^2)$$

for all $s \in \mathbb{R}$ and $t \in [s, s+4]$. Furthermore, given a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$, $R_0(\lambda)$ can be extended to a family of analytic operators for $\lambda \in \mathcal{S}$ as follows

$$\chi R_0(\lambda) \chi : H^s(\mathbb{R}^2) \rightarrow H^t(\mathbb{R}^2)$$

with the resolvent estimates

$$\|\chi R_0(\lambda) \chi\|_{\mathcal{L}(H^s, H^t)} \lesssim \left(\alpha(\lambda)^{-\frac{1}{2}} + \alpha(i\lambda)^{-\frac{1}{2}} \right) |\lambda|^{-2} (1 + |\lambda|)^{t-s} \left(e^{L(\Im \lambda)_-} + e^{L(\Re \lambda)_-} \right),$$

where $s \in \mathbb{R}$ and $t \in [s, s+4]$, $v_- := \max\{-v, 0\}$ and L is a constant larger than the diameter of the support of χ .

Appendix B. Generalized Gaussian random field

In this section, we introduce a more general class of random fields, which is called the generalized Gaussian random field.

Denote the space of test functions by $\mathcal{D}$, which is $C_0^\infty(\mathbb{R}^2; \mathbb{R})$ equipped with a locally convex topology. Denote the dual of $\mathcal{D}$ by $\mathcal{D}'$ which is the generalized function space. Given a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a complex-valued random field $g : \Omega \rightarrow \mathcal{D}'$ is called a generalized Gaussian random field if $\omega \mapsto \langle g(\omega), \phi \rangle$ is a complex-valued Gaussian random variable for all $\phi \in \mathcal{D}$. The expectation $\mathbb{E}g \in \mathcal{D}'$ and covariance $\text{Cov} g : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{C}$ are respectively defined by

$$\begin{aligned} \mathbb{E}g : \phi &\mapsto \mathbb{E}\langle g, \phi \rangle, \quad \phi \in \mathcal{D}, \\ \text{Cov} g : (\phi_1, \phi_2) &\mapsto \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}, \end{aligned}$$

where

$$\text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle) = \mathbb{E}[(\langle g, \phi_1 \rangle - \mathbb{E}\langle g, \phi_1 \rangle)(\langle g, \phi_2 \rangle - \mathbb{E}\langle g, \phi_2 \rangle)].$$

Define the covariance operator C_g by

$$\langle C_g \phi_1, \phi_2 \rangle = \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}.$$

Let $K_g(x, y)$ be the Schwarz kernel of C_g , i.e.

$$\langle C_g \phi_1, \phi_2 \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi_2(y) K_g(x, y) \phi_1(x) dx dy.$$

We consider the generalized Gaussian random fields with zero expectation defined as follows.

Definition B.1. A generalized Gaussian random field g with zero expectation is said to be microlocally isotropic of order $-m \in \mathbb{R}$ in $D \subset \mathbb{R}^2$, if its covariance operator C_g is a classical pseudodifferential operator of order $-m$ with the principal symbol $\mu(x)|\xi|^{-m}$, where $\mu(x) \in C_0^\infty(\mathbb{R}^2; \mathbb{C})$ is a complex-valued function with $\text{supp } \mu \subset\subset D$. Here μ is called the micro-correlation strength of g .

Let c_g be the symbol of the operator C_g , i.e.

$$C_g(\phi)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} c_g(x, \xi) \hat{\phi}(\xi) d\xi, \quad \phi \in \mathcal{D},$$

which gives

$$K_g(x, y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(x-y) \cdot \xi} c_g(x, \xi) d\xi.$$

The following lemma proved in [6] concerns the regularity of the GMIG random field.

Lemma B.2. *Let g be a GMIG random field of order $-m$ in D . Then $g \in W^{\frac{m-2}{2}-\epsilon, p}(D)$ almost surely for all $\epsilon > 0$ and $1 < p < \infty$.*

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