

Stability for inverse random source problems in electromagnetic and biharmonic waves

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Abstract

This paper is concerned with inverse random source problems for electromagnetic and biharmonic wave equations. The driven sources are assumed to be generalized microlocally isotropic Gaussian random fields such that the covariances are classical pseudo-differential operators. Uniqueness and stability are established for both inverse random source problems. The stability estimates consist of a Lipschitz type data discrepancy and a logarithmic stability, which decreases as the upper bound of wavenumbers increases. These increasing stability results reveal that ill-posedness can be overcome by using multi-wavenumber data. The analysis is based on integral equations and analytical continuation, which only requires multi-frequency Dirichlet data on the boundary in a finite interval and removes the limitation of data to be collected at all high wavenumbers. For the first time, the stability is established on the inverse source problems for both the Maxwell and biharmonic equations by Dirichlet boundary measurements.

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1. Introduction

Inverse random source problems in waves arise in many scientific and industrial areas such as biomedical imaging, geophysical exploration and radar and sonar when the surrounding environments are uncertain and random. In these situations, the sources of the waves should be described by random fields instead of deterministic functions, which cause the radiating wave fields to be random as well. Compared with the deterministic counterparts, the randomness makes it less meaningful to recover the full sources and the goal is to determine the statistics such as the mean and variance of the random sources, which are used to quantify uncertainties of the sources. Driven by the significant applications, inverse random source problems have been extensively investigated recently. Numerical and theoretical results can be found in [2–5,11,13,15–18,21] and references therein.

Inverse source problems for electromagnetic waves are considered as a basic mathematical tool for solving many inverse problems such as magnetoencephalography (MEG) in medical imaging and antenna design and synthesis. For results on deterministic inverse source problems in electromagnetic waves we refer the reader to [4]. For the inverse random source problems, uniqueness was studied in [16,18]. In these works, the source is assumed to be a generalized microlocally isotropic Gaussian (GMIG) random field proposed by [12], whose covariance operator is a classical pseudo-differential operator. Such Gaussian random fields include various classical processes such as white noise, fractional Brownian motions, Markov fields, and so on. For more details, we refer to [12,14]. On the other hand, from the computational point of view, a more challenging issue is the lack of stability for inverse problems, that is, a small variation of the data might lead to a huge error in the reconstruction. Thus, it is crucial to investigate the stability for inverse problems. However, to the best of our knowledge, little is known for the stability of inverse random source problems for Maxwell equations.

We also consider inverse random source problems for biharmonic waves. Such problems arise in thin plate elasticity and have received much attention due to their important applications in the control of destructive surface waves and the construction of ultrabroadband elastic cloaking devices [9,20]. Similar to Maxwell equations, uniqueness results are available for GMIG random sources in [15,16], but the corresponding stability estimates remain wide open. In this paper, we initialize the mathematical study of the stability estimates of the inverse source problems for the stochastic Maxwell and biharmonic equations driven by GMIG random sources from Dirichlet boundary measurements.

We develop a unified stability theory of the inverse random source problems for electromagnetic and biharmonic waves. It extends the previous approach developed in [19,21] for the Helmholtz and elastic wave equations to handle the more complicated Maxwell and biharmonic equations. For electromagnetic waves, the inverse problem is to reconstruct the matrix-valued principle symbol of the covariance operator of the random source. We show a uniqueness result and demonstrate that increasing stability estimate can be achieved by using only the tangential trace of the electric field on the boundary at multiple wavenumbers in a finite interval. For biharmonic waves, the inverse problem is to determine the principle symbol of the covariance operator of the random source. Uniqueness and the increasing stability are also achieved by using only multi-wavenumber Dirichlet boundary data in a finite interval. It is worth mentioning that we can remove the limitation that the wave field is required to be measured on a bounded domain at all high wavenumbers in [16,18]. The stability estimates for the two waves consist of a Lipschitz-type data discrepancy and a logarithmic stability. The logarithmic stability reveals the ill-posed nature of the inverse problem and is shown to decrease as the wavenumber of the data increases.

The stability illustrates that the ill-posedness of the inverse problems can be overcome and the inverse problems become more stable when multi-wavenumber data is used.

Our analysis consists of the following steps:

- First, we establish integral identities which connect the Fourier transform of the random source and the boundary data by using plane waves as test functions;
- Second, by taking the expectation of the correlated boundary integrals, the principle symbol of the random source can be expressed as a sum of the boundary data and a high frequency tail;
- Third, the stability is derived by investigating the analyticity of the data and applying a quantitative analytic continuation in complex theory to link the given data and the unavailable data at higher frequencies, which yields the logarithmic stability. The Dirichlet-to-Neumann maps are used to represent the Neumann data in terms of the Dirichlet data.

Recently, there have been some works on stability for inverse random source problems. A logarithmic stability estimate was derived in [17] for the stochastic Helmholtz equation driven by a white noise source. The proof relied on the Ito isometry of the white noise, which however does not hold for the generalized Gaussian random field. We studied in [21] the inverse random source problems for elastic waves, while in the current work we consider two different wave equations: the electromagnetic and biharmonic wave equations. Therefore, different techniques should be employed. For instance, since the double curl operator may not be elliptic unless the source is divergence-free, we make the assumption that the driven random source is divergence-free as in the deterministic setting [4], and then the Maxwell equation is reduced to the vector-valued Helmholtz equation. On the other hand, since the biharmonic wave equation is a fourth-order equation, the analysis becomes more involved.

This paper is organized as follows. In Section 2, we show the uniqueness and increasing stability of the inverse random source problem for Maxwell equations. Section 3 is devoted to the inverse random source problem for biharmonic waves. The uniqueness and increasing stability are obtained. In Appendix, we briefly introduce the definition and properties of the GMIG random field.

2. Electromagnetic waves

2.1. Problem formulation

We consider the time-harmonic Maxwell equations in a homogeneous medium:

$$\nabla \times \mathbf{E} - ik\mathbf{H} = 0, \quad \nabla \times \mathbf{H} + ik\mathbf{E} = \mathbf{J} \quad \text{in } \mathbb{R}^3, \quad (2.1)$$

where $k > 0$ is the wavenumber, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are both $\mathbb{C}^3$ -valued vector fields. The random electric current density $\mathbf{J}$ is a $\mathbb{R}^3$ -valued vector field, which is assumed to have a compact support in D , which is assumed to be a Lipschitz and bounded domain in $\mathbb{R}^3$. For $R > 0$, we denote $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ with boundary ∂B_R . Let R be large enough such that $D \subset \subset B_R$.

In this paper, we assume that the current density $\mathbf{J}$ is a generalized microlocally isotropic Gaussian random field. Then the source belongs to a Sobolev space with negative smooth index almost surely which should be treated as a distribution. Therefore, the Maxwell equations (2.1)

do not hold pointwisely and the weak Silver–Müller radiation condition is required to make the direct problem well-posed:

$$\lim_{r \rightarrow \infty} \int_{|x|=r} (\mathbf{H} \times \frac{x}{|x|} - \mathbf{E}) \cdot \boldsymbol{\phi} \, ds(x) = 0, \quad \text{for all } \boldsymbol{\phi} \in \mathcal{D}^3, \quad (2.2)$$

where $\mathcal{D}$ is the space of compactly supported functions equipped with standard local convex topology.

Eliminating the magnetic field $\mathbf{H}$ from (2.1), multiplying a test function $\mathbf{F} \in \mathcal{D}^3$, and integrating by parts over $\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} (\nabla \times (\nabla \times \mathbf{E}) - k^2 \mathbf{E}) \cdot \mathbf{F} \, dx = \int_{\mathbb{R}^3} ik \mathbf{J} \cdot \mathbf{F} \, dx. \quad (2.3)$$

Using $\nabla \times \nabla \times = -\Delta + \nabla \nabla \cdot$ and derivatives of distributions, the above identity becomes

$$\int_{\mathbb{R}^3} (-\Delta - k^2) \mathbf{E} \cdot \mathbf{F} \, dx - \int_{\mathbb{R}^3} (\nabla \cdot \mathbf{E})(\nabla \cdot \mathbf{F}) \, dx = \int_{\mathbb{R}^3} ik \mathbf{J} \cdot \mathbf{F} \, dx. \quad (2.4)$$

Moreover, it follows from the second equation in (2.1) that

$$\int_{\mathbb{R}^3} (\nabla \times \mathbf{H}) \cdot \nabla (\nabla \cdot \mathbf{F}) \, dx = -ik \int_{\mathbb{R}^3} \mathbf{E} \cdot \nabla (\nabla \cdot \mathbf{F}) \, dx + \int_{\mathbb{R}^3} \mathbf{J} \cdot \nabla (\nabla \cdot \mathbf{F}) \, dx,$$

which gives by $\nabla \cdot \nabla \times = 0$

$$\int_{\mathbb{R}^3} (\nabla \cdot \mathbf{E}) \cdot (\nabla \cdot \mathbf{F}) \, dx = \frac{i}{k} \int_{\mathbb{R}^3} \mathbf{J} \cdot \nabla (\nabla \cdot \mathbf{F}) \, dx. \quad (2.5)$$

In the deterministic cases, see e.g. [4], the source in Maxwell equations is usually assumed to satisfy the divergence free condition

$$\nabla \cdot \mathbf{J} = 0.$$

Thus, we introduce the space

$$\mathbb{B} = \left\{ \mathbf{U} \in (\mathcal{D}')^3 : \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \phi \, dx = 0 \text{ for all } \phi \in \mathcal{D} \right\}.$$

It can be seen that $\mathbb{B}$ consists of all divergence free vector fields in the sense of distribution. If $\mathbf{J} \in \mathbb{B}$, it follows from (2.3) and (2.5) that

$$\int_{\mathbb{R}^3} (-\Delta - k^2) \mathbf{E} \cdot \mathbf{F} \, dx = \int_{\mathbb{R}^3} ik \mathbf{J} \cdot \mathbf{F} \, dx,$$

which means that the following Helmholtz equation in the frequency domain holds in the distributional sense

$$-\Delta \mathbf{E} - k^2 \mathbf{E} = ik \mathbf{J} \quad \text{in } \mathbb{R}^3. \quad (2.6)$$

In this paper, we assume that $\mathbf{J}$ satisfies the following assumption.

Assumption 2.1. The random source $\mathbf{J}$ is a real-valued microlocally isotropic Gaussian random function of order $-n$ in the bounded Lipschitz domain D with zero expectation, where $n > 2$. Let $M > 0$ be a constant. The covariance operator has the matrix symbol $\mathbb{C}(x, \xi) = [c_{ij}]_{1 \leq i, j \leq 3}$ with the principal symbol $\mathbb{M}(x)|\xi|^{-n}$, $\mathbb{M} = [m_{ij}]_{1 \leq i, j \leq 3}$, satisfying (i) $|m_{ij}(x)| \leq M$ for $x \in D$; (ii) $|c_{ij}(x, \xi)| \leq M(1 + |\xi|)^{-n}$ for $\xi \in \mathbb{R}^3$ and $x \in D$; (iii) $|\mathbb{A}(x, \xi)| \leq M|\xi|^{-(n+1)}$ for $|\xi| \geq 1$ and $x \in D$ where $\mathbb{A}(x, \xi) = \mathbb{C}(x, \xi) - \mathbb{M}(x)|\xi|^{-n}$.

Under Assumption 2.1, according to [18, Theorem 2.2], the direct scattering problem (2.1)–(2.2) is well-posed almost surely with $\mathbf{E}$ belonging to a Sobolev space with positive smooth index. Moreover, it follows from the Helmholtz equation (2.6) that $\mathbf{E}$ has the following explicit representation

$$\mathbf{E}(x, k) = \frac{ik}{4\pi} \int_{B_R} \frac{e^{ik|x-y|}}{|x-y|} \mathbf{J}(y) dy. \quad (2.7)$$

Let $\mathbf{E} \times \mathbf{v}$ and $\mathbf{H} \times \mathbf{v}$ be the tangential trace of the electric field and the magnetic field, respectively. It is shown in [1, Section 2] that there exists a capacity operator T_M such that

$$\mathbf{H} \times \mathbf{v} = T_M(\mathbf{E} \times \mathbf{v}) \quad \text{on } \partial B_R, \quad (2.8)$$

which implies that $\mathbf{H} \times \mathbf{v}$ can be computed once $\mathbf{E} \times \mathbf{v}$ is available on ∂B_R . The transparent boundary condition (2.8) can be equivalently written as

$$(\nabla \times \mathbf{E}) \times \mathbf{v} = ik T_M(\mathbf{E} \times \mathbf{v}) \quad \text{on } \partial B_R. \quad (2.9)$$

We consider the following inverse random source problem by using random boundary measurements.

Problem 2.2. Let $\mathbf{J}$ be the random electric current density with the compact support D under Assumption 2.1. The inverse random source problem of electromagnetic waves is to determine the strength matrix $\mathbb{M}$ from the random tangential trace of the electric field $\mathbf{E}(x, k) \times \mathbf{v}$ for $x \in \partial B_R$.

2.2. Uniqueness for the inverse random source problem

In this section, we discuss the uniqueness of Problem 2.2.

Introduce the direction vector $\mathbf{d} = (d_1, d_2, d_3) \in \mathbb{S}^2$ and the polar vectors $\mathbf{p}_j, \mathbf{q}_j$ given by

$$\mathbf{p}_j = d_j \mathbf{d}, \quad \mathbf{q}_j = \mathbf{e}_j - \mathbf{p}_j, \quad 1 \leq j \leq 3,$$

where $\mathbf{e}_j$ is the unit vector in $\mathbb{R}^3$ with its j th entry being one. It is easy to see that $\mathbf{p}_j \perp \mathbf{q}_j$ and $\mathbf{p}_j \parallel \mathbf{d}$. Then we denote

$$I_{p,j}(\mathbf{d}, k) = \int_{B_R} e^{-ikx \cdot \mathbf{d}} \mathbf{p}_j \cdot \mathbf{J}(x) dx, \quad 1 \leq j \leq 3,$$

and

$$I_{q,l}(\mathbf{d}, k) = \int_{B_R} e^{-ikx \cdot \mathbf{d}} \mathbf{q}_l \cdot \mathbf{J}(x) dx, \quad 1 \leq l \leq 3.$$

For a matrix $\mathbb{P} = [p_{jl}]_{j,l=1,2,3}$ we introduce a rearrangement operator $\mathcal{A} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^9$ as follows

$$\mathcal{A}(\mathbb{P}) = (p_{11}, p_{12}, p_{13}, \dots, p_{31}, p_{32}, p_{33}),$$

which rearranges the entries of matrix $\mathbb{P}$ in rows into a vector.

Motivated by [16, Section 6], we consider the expectation of the correlated integrals and obtain

$$\begin{aligned} & \mathbb{E}[I_{p,j}(\mathbf{d}, (k + \tau)) \overline{I_{p,l}(\mathbf{d}, k)}] \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(k+\tau)\mathbf{d} \cdot \mathbf{y}} e^{ik\mathbf{d} \cdot \mathbf{z}} \mathbb{E}[\mathbf{p}_j \cdot \mathbf{J}(\mathbf{y}) \mathbf{p}_l \cdot \mathbf{J}(\mathbf{z})] d\mathbf{y} d\mathbf{z} \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(k+\tau)\mathbf{d} \cdot \mathbf{y}} e^{ik\mathbf{d} \cdot \mathbf{z}} \mathbb{E}[\mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top) \cdot \mathcal{A}(\mathbf{J}(\mathbf{y}) \mathbf{J}(\mathbf{z})^\top)] d\mathbf{y} d\mathbf{z} \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-ik\mathbf{d} \cdot (\mathbf{y} - \mathbf{z})} \mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top) \cdot \mathcal{A}(K_{\mathbf{J}}(\mathbf{y}, \mathbf{z})) d\mathbf{z} e^{-i\tau\mathbf{d} \cdot \mathbf{y}} d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top) \cdot \mathcal{A}(\mathbb{M}(\mathbf{y})) e^{-i\tau\mathbf{d} \cdot \mathbf{y}} d\mathbf{y} |k\mathbf{d}|^{-n} + O(|k|^{-n-1}) \\ &= \mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top) \cdot \mathcal{A}(\widehat{\mathbb{M}}(\tau\mathbf{d})) k^{-n} + O(k^{-n-1}), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & \mathbb{E}[I_{p,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(k+\tau)\mathbf{d} \cdot \mathbf{y}} e^{ik\mathbf{d} \cdot \mathbf{z}} \mathbb{E}[\mathbf{p}_j \cdot \mathbf{J}(\mathbf{y}) \mathbf{q}_l \cdot \mathbf{J}(\mathbf{z})] d\mathbf{y} d\mathbf{z} \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(k+\tau)\mathbf{d} \cdot \mathbf{y}} e^{ik\mathbf{d} \cdot \mathbf{z}} \mathbb{E}[\mathcal{A}(\mathbf{p}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(\mathbf{J}(\mathbf{y}) \mathbf{J}(\mathbf{z})^\top)] d\mathbf{y} d\mathbf{z} \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-ik\mathbf{d} \cdot (\mathbf{y} - \mathbf{z})} \mathcal{A}(\mathbf{p}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(K_{\mathbf{J}}(\mathbf{y}, \mathbf{z})) d\mathbf{z} e^{-i\tau\mathbf{d} \cdot \mathbf{y}} d\mathbf{y} \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^3} \mathcal{A}(\mathbf{p}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(\mathbb{M}(\mathbf{y})) e^{-i\tau \mathbf{d} \cdot \mathbf{y}} d\mathbf{y} |k\mathbf{d}|^{-n} + O(|k|^{-n-1}) \\
&= \mathcal{A}(\mathbf{p}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(\widehat{\mathbb{M}}(\tau \mathbf{d})) k^{-n} + O(k^{-n-1}),
\end{aligned} \tag{2.11}$$

where the matrix $\widehat{\mathbb{M}}$ is the Fourier transform of $\mathbb{M}$ given by $\widehat{\mathbb{M}}(\xi) = (\widehat{m}_{ij}(\xi))_{1 \leq i, j \leq 3}$ with $\widehat{m}_{ij}(\xi) = \int_{\mathbb{R}^3} \widehat{m}_{ij}(x) e^{-ix \cdot \xi} dx$. Here $K_J(\mathbf{y}, \mathbf{z})$ is the kernel of the covariance operator.

Similarly, we have

$$\begin{aligned}
&\mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{p,l}(\mathbf{d}, k)}] \\
&= \mathcal{A}(\mathbf{q}_j \mathbf{p}_l^\top) \cdot \mathcal{A}(\widehat{\mathbb{M}}(\tau \mathbf{d})) k^{-n} + O(k^{-n-1}),
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] \\
&= \mathcal{A}(\mathbf{q}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(\widehat{\mathbb{M}}(\tau \mathbf{d})) k^{-n} + O(k^{-n-1}).
\end{aligned} \tag{2.13}$$

On the other hand, by a direct calculation we obtain

$$\mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top + \mathbf{p}_j \mathbf{q}_l^\top + \mathbf{q}_j \mathbf{p}_l^\top + \mathbf{q}_j \mathbf{q}_l^\top) = \mathcal{A}(\mathbf{e}_j \mathbf{e}_l^\top),$$

which yields

$$\mathcal{A}(\mathbf{p}_j \mathbf{p}_l^\top + \mathbf{p}_j \mathbf{q}_l^\top + \mathbf{q}_j \mathbf{p}_l^\top + \mathbf{q}_j \mathbf{q}_l^\top) \cdot \mathcal{A}(\widehat{\mathbb{M}}(\tau \mathbf{d})) = \widehat{m}_{jl}(\tau \mathbf{d}). \tag{2.14}$$

Thus, according to (2.14), by adding (2.10)–(2.13) we derive

$$\begin{aligned}
\widehat{m}_{jl}(\tau \mathbf{d}) &= k^n \left(\mathbb{E}[I_{p,j}(\mathbf{d}, (k + \tau)) \overline{I_{p,l}(\mathbf{d}, k)}] \right. \\
&\quad + \mathbb{E}[I_{p,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] \\
&\quad + \mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{p,l}(\mathbf{d}, k)}] \\
&\quad \left. + \mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] \right) + O\left(\frac{1}{k}\right).
\end{aligned} \tag{2.15}$$

Now we connect the unknown strength $\mathbb{M}$ with boundary measurements. Introduce the test function $\mathbf{E}^{\text{inc}}(x) = \mathbf{q} e^{ikx \cdot \mathbf{d}}$. By eliminating $\mathbf{H}$, we obtain the decoupled Maxwell system for the electric field $\mathbf{E}$:

$$\nabla \times (\nabla \times \mathbf{E}) - k^2 \mathbf{E} = ik \mathbf{J} \quad \text{in } \mathbb{R}^3. \tag{2.16}$$

It is easy to verify that $\mathbf{E}^{\text{inc}}$ satisfies the homogeneous Maxwell equations in $\mathbb{R}^3$:

$$\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - k^2 \mathbf{E}^{\text{inc}} = 0. \tag{2.17}$$

Multiplying the both sides of (2.16) by $\mathbf{E}^{\text{inc}}$ with $\mathbf{q} = \mathbf{q}_l$, $l = 1, 2$, using the integration by parts over B_R and (2.17), we obtain

$$\begin{aligned}
& ik \int_{B_R} \mathbf{q}_l e^{-ik\mathbf{d} \cdot \mathbf{x}} \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} \\
&= - \int_{\partial B_R} \left(ik T_M(\mathbf{E}(x, k) \times \mathbf{v}) \cdot \mathbf{E}^{\text{inc}} + (\mathbf{E}(x, k) \times \mathbf{v}) \cdot (\nabla \times \mathbf{E}^{\text{inc}}) \right) ds,
\end{aligned}$$

which gives

$$ik I_{q,l}(\mathbf{d}, k) = - \int_{\partial B_R} \left(ik T_M(\mathbf{E}(x, k) \times \mathbf{v}) \cdot \mathbf{E}^{\text{inc}} + (\mathbf{E}(x, k) \times \mathbf{v}) \cdot (\nabla \times \mathbf{E}^{\text{inc}}) \right) ds. \quad (2.18)$$

On the other hand, since $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{p}_j \parallel \mathbf{d}$, we have

$$\begin{aligned}
I_{p,j}(\mathbf{d}, k) &= \int_{B_R} e^{-ik\mathbf{x} \cdot \mathbf{d}} \mathbf{p}_j \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} = c_j \int_{B_R} e^{-ik\mathbf{x} \cdot \mathbf{d}} \mathbf{d} \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} \\
&= -\frac{c_j}{ik} \int_{B_R} \nabla(e^{-ik\mathbf{x} \cdot \mathbf{d}}) \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} = \frac{c_j}{ik} \int_{B_R} (e^{-ik\mathbf{x} \cdot \mathbf{d}}) \nabla \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} = 0,
\end{aligned} \quad (2.19)$$

where c is the constant such that $\mathbf{p}_j = c_j \mathbf{d}$. Combining (2.15) and (2.18)–(2.19), we obtained the following formula which connects the expectation of the correlated boundary data and the strength of the random source

$$\widehat{m}_{jl}(\tau \mathbf{d}) = k^n \mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] + O\left(\frac{1}{k}\right). \quad (2.20)$$

Based on (2.20), we have the following uniqueness result.

Theorem 2.3. *Let $\mathbf{J}$ satisfy Assumption 2.1. Then for all $\mathbf{d} \in \mathbb{S}^2$ and $\tau > 0$ it holds almost surely that*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_T^{2T} k^n \left(\mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}] \right) dk = \widehat{m}_{jl}(\tau \mathbf{d}), \quad 1 \leq j, l \leq 3. \quad (2.21)$$

Remark 2.4. Notice that functions $\widehat{m}_{jl}$ are analytic since m_{jl} are compactly supported. Thus, with the help of analytic continuation, the above theorem implies that the strength $\mathbb{M} = [m_{jl}]_{1 \leq j, l \leq 3}$ can be uniquely reconstructed by (2.21) with $(\tau, \mathbf{d}) \in \Gamma$ where $\Gamma = \mathcal{I} \times \mathbb{O}$ is an open set of $\mathbb{R}^+ \times \mathbb{S}^2$. Here $\mathcal{I}$ is an open interval in $\mathbb{R}^+$ and $\mathbb{O}$ is an open set of $\mathbb{S}^2$.

Remark 2.5. The assumption $\mathbf{J} \in \mathbb{B}$ is a technical assumption which is sufficient for the study of the inverse problem. However, it is not known if this condition is also necessary.

2.3. Stability for the inverse random source problem

In this section, we derive a stability estimate for Problem 2.2. Define a function space

$$\mathcal{C}_s := \{\mathbb{H} = [h_{jl}]_{1 \leq j, l \leq 3} \in C_0^\infty(D; \mathbb{R}^{3 \times 3}) : \|h_{jl}\|_{H^s(\mathbb{R}^3)} \leq Q, 1 \leq j, l \leq 3\}$$

with constants $Q, s > 0$. We introduce the correlation boundary data

$$\epsilon_{jl}(k, \tau, \mathbf{d}) = k^n \mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}]$$

and

$$\epsilon_{jl} = \sup_{k \in (0, K), \mathbf{d} \in \mathbb{S}^2, \eta \in (0, 1)} |\epsilon_{jl}(k, \eta k, \mathbf{d})|, \quad (2.22)$$

where K is an arbitrary positive constant.

Theorem 2.6. Assume that $\mathbf{J}$ satisfies Assumption 2.1 with $\mathbf{J} \in \mathbb{B}$ and $\mathbb{M} \in \mathcal{C}_s$. There holds the following stability estimate

$$\|m_{jl}\|_{L^2(\mathbb{R}^3)}^2 \lesssim K^{\alpha_1} \epsilon_{jl}^2 + \frac{1}{K^{\alpha_2} E_{jl}^{\alpha_3}}, \quad 1 \leq j, l \leq 3, \quad (2.23)$$

where $E_{jl} = |\log \epsilon_{jl}|$ with ϵ_{jl} being defined in (2.22), $\alpha_1 = \frac{6}{3+2s}$, $\alpha_2 = \frac{8s}{3(2s+3)}$, and $\alpha_3 = \frac{s}{2s+3}$.

Hereafter, the notation “ $a \lesssim b$ ” stands for $a \leq Cb$ where $C > 0$ is a generic constant. We also use the notation c to denote a positive constant. The values of the constants c and C may change step by step but should be clear from the context.

Remark 2.7. The stability estimate (2.23) consists of two parts: a Lipschitz stability and a logarithmic stability, where the latter diminishes as the upper bound K of the wavenumber increases.

In what follows, we prove Theorem 2.6. The strategy is to show that the data $\epsilon_{jl}(k, \tau, \mathbf{d})$ is analytic with respect to complex wavenumber k and apply a quantitative analytic continuation in complex theory.

Noting the explicit representation (2.7) of the electric field, from direct calculations, it is easy to see that the right hand side of the identity (2.20) can be written as a sum of integrals of the following form

$$I(\mathbf{d}, k) := k^{n-1} \int_{\partial B_R} \int_{\partial B_R} \int_D \int_D K(\tilde{x}, \tilde{y}) P(x, y, \tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} dx dy, \quad (2.24)$$

where K is some entry in the kernel matrix $K_{\mathbf{J}}$ of the covariance operator for the random source and P has the following form

$$k^{2-|\alpha|-|\beta|} \partial_x^\alpha G(x, \tilde{x}) \partial_y^\beta G(y, \tilde{y}) e^{-ik(x \cdot \mathbf{d}_1 + y \cdot \mathbf{d}_2)},$$

where $\alpha, \beta \in \mathbb{N}^3$ with $|\alpha|, |\beta| \leq 1$. Here $G(x, y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}$. Moreover, we have

$$|\nabla_x G(x, y)|, |\nabla_y G(x, y)| \lesssim \frac{|k|e^{|k||x-y|}}{|x-y|} + \frac{e^{|k||x-y|}}{|x-y|^2}. \quad (2.25)$$

To estimate $I(d, k)$, the following lemma is useful which is a corollary of Lemma A.3.

Lemma 2.8. *Let the random function $\mathbf{J}$ satisfy Assumption 2.1. Each entry of the covariance matrix kernel function $K_{\mathbf{J}} = [K_{ij}]_{1 \leq i, j \leq 3}$ has the following form:*

(i) *If $a < n - 2 \leq a + 1$ with $a = 1, 2, 3, \dots$,*

$$K_{ij} = cm_{ij}(x)|x - y|^{n-3} + F_n(x, y)$$

where c is a constant dependent on n and $F_n(x, y) \in C^{a, \alpha}(\mathbb{R}^3 \times \mathbb{R}^3)$ with $\alpha \in (0, n - 2 - a)$.

(ii) *If $2 < n < 3$,*

$$K_{ij} = cm_{ij}(x)|x - y|^{n-3} + F_n(x, y)$$

where c is a constant dependent on n and $F_m(x, y) \in C^{0, \alpha}(\mathbb{R}^3 \times \mathbb{R}^3)$ with $\alpha \in (0, n - 2)$.

(iii) *If $n = 3$,*

$$K_{ij} = cm_{ij}(x) \log |x - y| + F_n(x, y)$$

where c is a constant dependent on $F_n(x, y) \in C^{0, \alpha}(\mathbb{R}^3 \times \mathbb{R}^3)$ with $\alpha \in (0, 1)$.

For all of the above three cases, we have

$$\|F_n\|_{L^\infty(D \times D)} \lesssim M.$$

Denote a sectorial domain by $\mathcal{S} = \{z \in \mathbb{C} : |\arg z| < \pi/4\}$. We show that I is bounded and analytic with respect to $k = k_1 + ik_2 \in \mathcal{S}$.

We first consider the case when $n > 2$ and $n \neq 3$. We have from Lemma 2.8 that the kernel $K(\tilde{x}, \tilde{y})$ has the representation

$$K(\tilde{x}, \tilde{y}) = cm(\tilde{x})|\tilde{x} - \tilde{y}|^{n-3} + F_n(\tilde{x}, \tilde{y}),$$

where $m(\tilde{x})$ is some entry in the matrix $\mathbb{M}(\tilde{x})$. Therefore, noticing that $D \cap \partial B_R = \emptyset$, the following estimates

$$\int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |K(\tilde{x}, \tilde{y})P(x, y, \tilde{x}, \tilde{y}; k)| d\tilde{x} d\tilde{y} ds(x) ds(y) < \infty$$

and

$$\int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |K(\tilde{x}, \tilde{y})\partial_k P(x, y, \tilde{x}, \tilde{y}; k)| d\tilde{x} d\tilde{y} ds(x) ds(y) < \infty$$

hold for $k \in \mathcal{S}$, which gives that $I(\mathbf{d}, k)$ is analytic with respect to $k \in \mathcal{S}$ by the dominated convergence theorem. Moreover, we have for $k \in \mathcal{S}$ that

$$\begin{aligned} |I(\mathbf{d}, k)| &\lesssim |k|^{n+1-|\alpha|-|\beta|} e^{c|k|} \\ &\quad \times \int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |\partial^\alpha G(x, \tilde{x}) \partial^\beta G(y, \tilde{y})| (m(\tilde{x}) |\tilde{x} - \tilde{y}|^{n-3} + M) d\tilde{x} d\tilde{y} ds(x) ds(y) \\ &\lesssim |k|^{n+1-|\alpha|-|\beta|} e^{c|k|} \\ &\quad \times \int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |\partial^\alpha G(x, \tilde{x}) \partial^\beta G(y, \tilde{y})| (|\tilde{x} - \tilde{y}|^{n-3} + 1) d\tilde{x} d\tilde{y} ds(x) ds(y). \end{aligned} \quad (2.26)$$

Then from (2.25) we obtain the following estimate

$$|I(\mathbf{d}, k)| \lesssim |k|^{n-1} e^{c|k|} (1 + |k|)^2 \lesssim e^{c|k|} \quad (2.27)$$

by $n > 2$.

For the case $n = 3$, as $K(\tilde{x}, \tilde{y}) = cm(\tilde{x}) \log |\tilde{x} - \tilde{y}| + F_n(\tilde{x}, \tilde{y})$ by Lemma 2.8, we can derive the estimate (2.27) in a similar manner.

Based on the analysis above, we derive the analyticity and an upper bound for $I(\mathbf{d}, k)$ with respect to the complex wavenumber $k \in \mathcal{S}$. In summary, we arrive at the following lemma.

Lemma 2.9. *The integral $I(\mathbf{d}, k)$ is analytic with respect to $k \in \mathcal{S}$. Furthermore, the following estimate*

$$|I(\mathbf{d}, k)| \lesssim e^{c|k|}$$

holds for $k \in \mathcal{S}$.

According to (2.15), we have

$$\widehat{m_{jl}}(\tau \mathbf{d}) = \epsilon(k, \tau, \mathbf{d}) + O\left(\frac{1}{k}\right),$$

where

$$\epsilon(k, \tau, \mathbf{d}) = k^n \mathbb{E}[I_{q,j}(\mathbf{d}, (k + \tau)) \overline{I_{q,l}(\mathbf{d}, k)}]. \quad (2.28)$$

Taking $\tau = \eta k$ with $\eta \in (0, 1)$, we have the inequality

$$|\widehat{m_{jl}}(\xi)| \lesssim \sup_{\eta \in (0, 1), \mathbf{d} \in \mathbb{S}^2} |\tilde{\epsilon}(k, \eta, \mathbf{d})| + \frac{1}{k}, \quad |\xi| < k, \quad (2.29)$$

where

$$\tilde{\epsilon}(k, \eta, \mathbf{d}) := \epsilon(k, \eta k, \mathbf{d})$$

As a corollary of Lemma 2.9, we have the following lemma.

Lemma 2.10. *It holds that $\epsilon(k, \tau, \mathbf{d})$ is analytic with the following estimate for $k \in \mathcal{S}$*

$$|\tilde{\epsilon}(k, \eta, \mathbf{d})| \lesssim e^{c|k|}.$$

The following analytic continuation principle [8, Section 3] is useful in the subsequent analysis.

Lemma 2.11. *Let $U(z)$ be an analytic function in $\mathcal{S}$ and continuous in $\bar{\mathcal{S}}$. If*

$$\begin{cases} |U(z)| \leq \epsilon, & z \in (0, L], \\ |U(z)| \leq V, & z \in \mathcal{S}, \\ |U(z)| = 0, & z = 0, \end{cases}$$

with constants $\epsilon, L, V > 0$, then there exists a function $\mu(z)$ satisfying

$$\begin{cases} \mu(z) \geq 1/2, & z \in (L, 2^{1/4}L), \\ \mu(z) \geq \pi^{-1}((z/L)^4 - 1)^{-1/2}, & z \in (2^{1/4}L, \infty) \end{cases}$$

such that

$$|U(z)| \leq V\epsilon^{\mu(z)}, \quad \forall z \in (L, \infty).$$

Denote

$$\epsilon^2 = \sup_{k \in (0, K), \mathbf{d} \in \mathbb{S}^2, \eta \in (0, 1)} |\epsilon^2(k, \eta k, \mathbf{d})|, \quad (2.30)$$

where $\epsilon(k, \tau, \mathbf{d})$ is defined in (2.28) and $K > 0$ is a positive constant. Using Lemma 2.9–2.11, we have the following important lemma, which provides an estimate of the data at high wavenumbers by the data at small wavenumbers.

Lemma 2.12. *There exists a positive constant $c > 0$ such that the following estimate holds uniformly for $\eta \in (0, 1)$ and $\mathbf{d} \in \mathbb{S}^2$*

$$|\tilde{\epsilon}^2(k, \eta, \mathbf{d})| \lesssim \epsilon^{2\mu(k)} e^{ck}, \quad k \in (K, \infty)$$

with the function $\mu(k)$ satisfying

$$\begin{cases} \mu(k) \geq 1/2, & k \in (K, 2^{1/4}K), \\ \mu(k) \geq \pi^{-1}((k/K)^4 - 1)^{-1/2}, & k \in (2^{1/4}K, \infty). \end{cases} \quad (2.31)$$

Proof. We first extend $|\tilde{\epsilon}(k, \eta, \mathbf{d})|^2$ from $k > 0$ to $k \in \mathcal{S}$. To this end, let

$$U(k) = (-1)^n \tilde{\epsilon}(k, \eta, \mathbf{d}) \tilde{\epsilon}(-k, \eta, \mathbf{d}), \quad k \in \mathcal{S},$$

which is analytic and continuous for $k \in \bar{S}$. Noting $\overline{u(x, k)} = u(x, -k)$, we have $U(k) = |\bar{\epsilon}(k, \eta, \mathbf{d})|^2$ for $k > 0$, which is useful for the subsequent application of analytic continuation.

It follows from Lemma 2.9 for $k \in S$ that

$$|U(k)| \lesssim e^{c|k|},$$

which yields $|e^{-ck}U(k)|$ is bounded. Obviously, for $k \in (0, K]$, one has $|e^{-ck}U(k)| \lesssim \epsilon^2$. Applying Lemma 2.11 to $e^{-ck}U(k)$ gives

$$|e^{-ck}U(k)| \lesssim \epsilon^{2\mu(k)}, \quad k \in (K, \infty)$$

where $\mu(k)$ satisfies (2.31). Thus, we have

$$|\tilde{\epsilon}^2(k, \eta, \mathbf{d})| \lesssim \epsilon^{2\mu(k)} e^{ck}, \quad k \in (K, \infty),$$

which completes the proof. $\square$

The proof of Theorem 2.6 follows the arguments in [19, Theorem 4.1] by applying Lemma 2.12 and an analytic continuation principle. The proof is presented in Appendix for the reader's convenience.

3. Biharmonic waves

3.1. Problem formulation

In this section, we consider the biharmonic wave equation in two and three dimensions

$$\Delta^2 u - k^4 u = f \quad \text{in } \mathbb{R}^d, \quad (3.1)$$

where $d = 2, 3$, and $k > 0$ is the wavenumber. The scattered field u and its Laplacian Δu are required to satisfy the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} (\partial_r u - iku) = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} (\partial_r (\Delta u) - ik(\Delta u)) = 0 \quad r = |x|. \quad (3.2)$$

The source f is assumed to be a generalized microlocally isotropic Gaussian random field of order $-n$ such that its covariance operator is a classical pseudo-differential operator with principle symbol $h(x)|\xi|^{-n}$, where h is called the micro-correlation strength of the random source. In this work, we require the random source f to satisfy the following assumption:

Assumption 3.1. The random source f is a real-valued microlocally isotropic Gaussian random function of order $-n$ in the bounded Lipschitz D such that $n > d - 1$ with zero expectation. Its covariance operator has the symbol $c_f(x, \xi) = h(x)|\xi|^{-n} + a_f(x, \xi)$ with principal symbol $h(x)|\xi|^{-n}$ which satisfies the conditions: (i) $|h(x)| \leq M$ for $x \in D$; (ii) $|c_f(x, \xi)| \leq M(1 + |\xi|)^{-n}$ for $\xi \in \mathbb{R}^d$ and $x \in D$; (iii) $|a_f(x, \xi)| \leq M|\xi|^{-(n+1)}$ for $|\xi| \geq 1$ and $x \in D$.

Under Assumption 3.1, the direct scattering problem (3.1)–(3.2) is well-posed (cf. [15, Theorem 3.2]). Moreover, the solution has the following explicit form

$$u(x) = \int_{B_R} \Phi(x, y) f(y) dy, \quad (3.3)$$

where $\Phi(x, y) = \frac{1}{2k^2} (G_k(x, y) - G_{ik}(x, y))$ is the fundamental solution of the biharmonic waves. Here

$$G_\lambda(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(\lambda|x-y|), & d=2, \\ \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}, & d=3. \end{cases}$$

It is shown in [7, (18)] that there exists Dirichlet-to-Neumann operators T_1 and T_2 such that

$$\partial_\nu u = T_1(u, \Delta u), \quad \partial_\nu \Delta u = T_2(u, \Delta u) \quad \text{on } \partial B_R, \quad (3.4)$$

which means that $\partial_\nu u$ and $\partial_\nu \Delta u$ can be computed from u and Δu on ∂B_R .

We consider the following inverse random source problem from Dirichlet boundary data.

Problem 3.2. Let f be the random source with the compact support D under Assumption 3.1. The inverse random source problem of biharmonic waves is to determine the strength h from the Dirichlet boundary measurements $u, \Delta u$ for $x \in \partial B_R$.

3.2. Uniqueness for the inverse random source problem

Let $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}$. Consider the covariance of the Fourier transform of f , which can be expressed by

$$\begin{aligned} \mathbb{E}[\hat{f}(k\mathbf{d}_1) \hat{f}(k\mathbf{d}_2)] &= \mathbb{E} \left(\int_{B_R} \int_{B_R} f(x) f(y) e^{-ik\mathbf{d}_1 \cdot x} e^{-ik\mathbf{d}_2 \cdot y} dx dy \right) \\ &= \int_{B_R} \int_{B_R} K_f(x, y) e^{-ik\mathbf{d}_1 \cdot x} e^{-ik\mathbf{d}_2 \cdot y} dx dy \\ &= \int_{B_R} \int_{B_R} \int_{\mathbb{R}^d} e^{-i(k\mathbf{d}_1 \cdot x + k\mathbf{d}_2 \cdot y)} e^{i(x-y) \cdot \zeta} c(x, \zeta) d\zeta dx dy \\ &= \int_{B_R} \int_{\mathbb{R}^d} e^{ix \cdot (\zeta - k\mathbf{d}_1)} c(x, \zeta) \delta(\zeta + k\mathbf{d}_2) d\zeta dx = \int_{B_R} e^{-ix \cdot (k\mathbf{d}_1 + k\mathbf{d}_2)} c(x, -k\mathbf{d}_2) dx \\ &= \int_{B_R} e^{-ix \cdot (k\mathbf{d}_1 + k\mathbf{d}_2)} h(x) k^{-n} dx + O(k^{-(n+1)}) \\ &= k^{-n} \hat{h}(k\mathbf{d}_1 + k\mathbf{d}_2) + O(k^{-(n+1)}). \end{aligned}$$

Therefore, we obtain

$$\hat{h}(k(\mathbf{d}_1 + \mathbf{d}_2)) = k^n \mathbb{E}(\hat{f}(k\mathbf{d}_1)\hat{f}(k\mathbf{d}_2)) + O\left(\frac{1}{k}\right), \quad (3.5)$$

where $\hat{f}$ and $\hat{h}$ are the Fourier transforms of f and h , respectively.

Next, we connect the covariance $\mathbb{E}[\hat{f}(k\mathbf{d}_1)\hat{f}(k\mathbf{d}_2)]$ to boundary measurements. To this end, multiplying both sides of the equation (3.1) by $e^{ikx \cdot \mathbf{d}_j}$, integrating by parts over B_R and noting that $e^{ikx \cdot \mathbf{d}_j}$ satisfies the homogeneous biharmonic wave equation

$$(\Delta^2 - k^4)e^{ikx \cdot \mathbf{d}_j} = 0,$$

we obtain

$$\begin{aligned} \hat{f}(k\mathbf{d}_j) &= \int_{B_R} f(x) e^{-ikx \cdot \mathbf{d}_j} dx \\ &= \int_{\partial B_R} \left(\partial_\nu (\Delta u(x, k)) e^{-ikx \cdot \mathbf{d}_j} - \Delta u(x, k) \partial_\nu e^{-ikx \cdot \mathbf{d}_j} \right) ds \\ &\quad + \int_{\partial B_R} \left(\partial_\nu u(x, k) \Delta e^{-ikx \cdot \mathbf{d}_j} - u(x, k) \partial_\nu (\Delta e^{-ikx \cdot \mathbf{d}_j}) \right) ds, \quad j = 1, 2. \end{aligned}$$

Then we have

$$\begin{aligned} &\mathbb{E}[\hat{f}(k\mathbf{d}_1)\hat{f}(k\mathbf{d}_2)] \\ &= k^4 \sum_{|\alpha|, |\beta| \leq 1} \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_\nu^\alpha u(x, k) \partial_\nu^\beta u(y, k)] e^{-ik(\mathbf{d}_1 + \mathbf{d}_2)} \\ &\quad \times (-ik\mathbf{d}_1 \cdot \hat{x})^{1-|\alpha|} (-ik\mathbf{d}_2 \cdot \hat{y})^{1-|\beta|} ds(x) ds(y) \\ &\quad - 2k^2 \sum_{|\alpha|, |\beta| \leq 1} \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_\nu^\alpha \Delta u(x, k) \partial_\nu^\beta u(y, k)] e^{-ik(\mathbf{d}_1 + \mathbf{d}_2)} \\ &\quad \times (-ik\mathbf{d}_1 \cdot \hat{x})^{1-|\alpha|} (-ik\mathbf{d}_2 \cdot \hat{y})^{1-|\beta|} ds(x) ds(y) \\ &\quad + \sum_{|\alpha|, |\beta| \leq 1} \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_\nu^\alpha \Delta u(x, k) \partial_\nu^\beta \Delta u(y, k)] e^{-ik(\mathbf{d}_1 + \mathbf{d}_2)} \\ &\quad \times (-ik\mathbf{d}_1 \cdot \hat{x})^{1-|\alpha|} (-ik\mathbf{d}_2 \cdot \hat{y})^{1-|\beta|} ds(x) ds(y) \\ &=: \sum_{j=1}^3 I_j(k, \mathbf{d}_1, \mathbf{d}_2). \end{aligned} \quad (3.6)$$

Combining (3.5) and (3.6) we obtain

$$\hat{h}(k(\mathbf{d}_1 + \mathbf{d}_2)) = k^n \sum_{j=1}^3 I_j(k, \mathbf{d}_1, \mathbf{d}_2) + O\left(\frac{1}{k}\right). \quad (3.7)$$

Theorem 3.3. *Let f satisfy Assumption 3.1. Then f is uniquely determined by the correlation near-field data $I_j(k, \mathbf{d}_1, \mathbf{d}_2)$, $j = 1, 2, 3$, where $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}$, $k \in [K_0, +\infty)$. Here K_0 is a positive constant.*

Proof. Let $K_0 > 0$ be an arbitrary positive constant. It suffices to prove that $I_j(k, \mathbf{d}_1, \mathbf{d}_2) = 0$, $j = 1, 2, 3$, for all $k > K_0$ and $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}$ gives $\hat{h}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. It follows from $\{\xi : \xi = k(\mathbf{d}_1 + \mathbf{d}_2)\} = \{\xi : |\xi| \leq 2k\}$ and (3.7) that, for any $\xi_0 \in \mathbb{R}^d$, there exists $k > 0$ sufficiently large such that

$$\hat{h}(\xi_0) = O\left(\frac{1}{k}\right).$$

Letting $k \rightarrow \infty$ yields $\hat{h}(\xi_0) = 0$, which implies $h \equiv 0$ by taking the inverse Fourier transform. The proof is complete. $\square$

Remark 3.4. In fact, with the help of analytic continuation, f can be uniquely determined from the data $\{I_j(k_j, \mathbf{d}_1, \mathbf{d}_2) : j \in \mathbb{N}_+, \mathbf{d}_1 \in \Gamma_1, \mathbf{d}_2 \in \Gamma_2\}$, where Γ_1, Γ_2 are open sets of $\mathbb{S}^{d-1}$ such that $\Gamma_1 \cap (-\Gamma_2) \neq \emptyset$ and $\{k_n : n \in \mathbb{N}_+\} \subset [K_0, +\infty)$ is a sequence which goes to infinity.

3.3. Stability for the inverse random source problem

In this section, we consider the stability for Problem 3.2. Define the boundary measurements

$$\varepsilon^2(k, \mathbf{d}_1, \mathbf{d}_2) = k^{2n} \sum_{j=1}^3 |I_j(k, \mathbf{d}_1, \mathbf{d}_2)|^2$$

and

$$\varepsilon^2 = \sup_{k \in (0, K], \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}} \varepsilon^2(k, \mathbf{d}_1, \mathbf{d}_2), \quad (3.8)$$

where K is a positive constant. Define the function space $\mathcal{H}_s = \{h \in C_0^\infty(D) : \|h\|_{H^s(\mathbb{R}^d)} \leq Q\}$ with $s > 0$, $Q > 0$. The following stability for inverse random source problem is the main result in this section.

Theorem 3.5. *Assume that f satisfies Assumption 3.1 and the strength $h \in \mathcal{H}_s$. There holds the following stability estimate*

$$\|h\|_{L^2(\mathbb{R}^d)}^2 \lesssim K^{\beta_1} \varepsilon^2 + \frac{1}{K^{\beta_2} E^{\beta_3}}, \quad (3.9)$$

where $E = |\log \varepsilon|$ with ε being defined in (3.8), $\beta_1 = \frac{2d}{d+2s}$, $\beta_2 = \frac{8s}{3(2s+d)}$, and $\beta_3 = \frac{s}{2s+d}$.

We prove Theorem 3.5. From (3.7) and noting

$$\{k(\mathbf{d}_1 + \mathbf{d}_2) : \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}\} = \{\xi \in \mathbb{R}^d : |\xi| \leq 2k\},$$

we derive the following estimate for all $|\xi| \leq 2k$

$$|\hat{h}(\xi)|^2 \lesssim k^{2n} \sum_{j=1}^3 |I_j(k, \mathbf{d}_1, \mathbf{d}_2)|^2 + \frac{1}{k^2}.$$

We prove the stability following the strategy for Maxwell equations, that is, we show the analyticity and derive estimates of the data with respect to complex wavenumber $k \in \mathcal{S}$. To achieve this goal, we study one term in the sum of I_1 , which is

$$I(k, \mathbf{d}_1, \mathbf{d}_2) = k^{n+4} \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_{\mathbf{v}} u(x, k) \partial_{\mathbf{v}} u(y, k)] e^{-ik(\mathbf{d}_1 + \mathbf{d}_2)} \mathrm{d}s(x) \mathrm{d}s(y).$$

The analysis of the other terms follows in a similar manner.

For $n > d - 1$ and $n \neq d$, as before, by Lemma A.3 we have

$$K_f(x, y) = ch(x)|x - y|^{n-d} + F_n(x, y).$$

Therefore, letting

$$L(x, y, \tau, t; k) = \partial_{\mathbf{v}(x)} \Phi(x, \tau) \partial_{\mathbf{v}(y)} \Phi(y, t) e^{-ik(x \cdot \mathbf{d}_1 + y \cdot \mathbf{d}_2)},$$

we have

$$\int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |K_f(\tau, t) L(x, y, \tau, t; k)| \mathrm{d}\tau \mathrm{d}t \mathrm{d}s(x) \mathrm{d}s(y) < \infty, \quad (3.10)$$

$$\int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |K_f(\tau, t) \partial_k L(x, y, \tau, t; k)| \mathrm{d}\tau \mathrm{d}t \mathrm{d}s(x) \mathrm{d}s(y) < \infty. \quad (3.11)$$

As a consequence, the derivative can be taken under the integral which yields the analyticity for $k = k_1 + ik_2 \in \mathcal{S}$. Moreover, we derive the following bound

$$\begin{aligned} |I| &\lesssim k^{n+4} e^{c|k|} \int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |\partial_{\mathbf{v}(x)} \Phi(x, \tau) \partial_{\mathbf{v}(y)} \Phi(y, t)| (h(\tau)|\tau - t|^{n-d} + M) \mathrm{d}\tau \mathrm{d}t \mathrm{d}s(x) \mathrm{d}s(y) \\ &\lesssim k^{n+4} e^{c|k|} \int_{\partial B_R} \int_{\partial B_R} \int_D \int_D |\partial_{\mathbf{v}(x)} \Phi(x, \tau) \partial_{\mathbf{v}(y)} \Phi(y, t)| (|\tau - t|^{n-d} + 1) \mathrm{d}\tau \mathrm{d}t \mathrm{d}s(x) \mathrm{d}s(y). \end{aligned} \quad (3.12)$$

In the following analysis, we focus on the two-dimensional case, which is $d = 2$. We start with studying the Green function in two dimensions which is a sum of Hankel and modified

Hankel functions. The analysis for $d = 3$ will follow in a similar manner. In fact, since the Green function in three dimensions has an explicit exponential form, the analysis for $d = 3$ is easier.

From [10, (1.14)], we have the following integral form of the Hankel function

$$H_0^{(1)}(z) = C e^{iz} \int_0^\infty e^{-s} s^{-1/2} (s/2 - iz)^{-1/2} ds, \quad (3.13)$$

where C is a constant, and the branch cut of $\sqrt{z}$ is chosen such that $\sqrt{z}$ is analytic for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\sqrt{1} = 1$. Notice that $\Phi_\lambda(x, y) = \frac{1}{4} H_0^{(1)}(\lambda|x - y|)$. For $\lambda \in \Omega := \mathbb{C} \setminus (-\infty, 0]i$, we have $\sqrt{s/2 - i\lambda|x - y|} \in \mathbb{C} \setminus (-\infty, 0]$ for any $x, y \in \mathbb{R}^2$ and $s > 0$. Moreover, from (3.13) we have the estimate

$$|\Phi_\lambda(x, y)| \lesssim \frac{e^{(\Im \lambda)|x-y|}}{\sqrt{\alpha(\lambda)|x-y|}} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt \lesssim \frac{e^{(\Im \lambda)|x-y|}}{\sqrt{\alpha(\lambda)|x-y|}}, \quad \lambda \in \Omega, \quad (3.14)$$

where $a_- := \max\{-a, 0\}$ and $\alpha(\lambda)$ is given by

$$\alpha(\lambda) = \begin{cases} |\Re \lambda|, & \Im \lambda \leq 0, \\ |\lambda|, & \Im \lambda > 0. \end{cases}$$

Thus, noting $\Phi(x, y) = \frac{1}{2k^2} (G_k(x, y) - G_{ik}(x, y))$, we obtain

$$|\partial_{\nu(y)} \Phi(x, y)| \lesssim \frac{1}{k^2} \left(\frac{|k| e^{|k||x-y|}}{|k(x-y)|^{\frac{1}{2}}} + \frac{e^{|k||x-y|}}{|k|^{\frac{1}{2}} |x-y|^{\frac{3}{2}}} \right), \quad k \in \mathcal{S}, x \in D, y \in \partial B_R.$$

Then in a similar manner as the derivation of (2.27), we obtain

$$|I| \lesssim |k|^{n-1} e^{c|k|} (1 + |k|)^2. \quad (3.15)$$

We can also obtain the estimate $|I| \lesssim k^n e^{c|k|} (1 + |k|)^2$ for $d = 3$ by investigating the more explicit exponential fundamental solution in a similar way. In summary, by $n > d - 1$, we obtain the following estimate

$$|I| \lesssim |k|^{n+d-3} e^{c|k|} (1 + |k|)^2 \lesssim e^{c|k|}, \quad k \in \mathcal{S}, d = 2, 3.$$

Now we extend $|I(k, \mathbf{d}_1, \mathbf{d}_2)|^2$ to the complex plane. Because the radiating solution to the biharmonic wave equation satisfies $u(x, k) = u(x, ik)$, we define

$$\begin{aligned} \varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2) &= k^{2(n+8)} \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_\nu u(x, k) \partial_\nu u(y, k)] e^{ik(\mathbf{d}_1 + \mathbf{d}_2)} ds(x) ds(y) \\ &\quad \times \int_{\partial B_R} \int_{\partial B_R} \mathbb{E}[\partial_\nu u(x, ik) \partial_\nu u(y, ik)] e^{-ik(\mathbf{d}_1 + \mathbf{d}_2)} ds(x) ds(y), \quad k \in \mathcal{S}. \end{aligned}$$

Note that for $k > 0$ we have

$$\varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2) = |I(k, \mathbf{d}_1, \mathbf{d}_2)|^2.$$

By this extension, we have that $\varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2)$ is analytic for $k \in \mathcal{S}$ with the estimate

$$|\varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2)| \lesssim e^{2c|k|}.$$

Denoting

$$\varepsilon_I^2 = \sup_{\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{S}^{d-1}, k \in (0, K]} \varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2),$$

and following the argument of applying analytic continuation for proving Lemma 2.12, we have the following lemma.

Lemma 3.6. *We have the estimate*

$$\varepsilon_I^2(k, \mathbf{d}_1, \mathbf{d}_2) \lesssim \varepsilon_I^{2\mu(k)} e^{2ck}, \quad k \in (K, \infty)$$

with the function $\mu(k)$ satisfying

$$\begin{cases} \mu(k) \geq 1/2, & k \in (K, 2^{1/4}K), \\ \mu(k) \geq \pi^{-1}((k/K)^4 - 1)^{-1/2}, & k \in (2^{1/4}K, \infty). \end{cases} \quad (3.16)$$

Since the analysis for $|I(k, \mathbf{d}_1, \mathbf{d}_2)|^2$ can be carried over to the sum $|I_j(k, \mathbf{d}_1, \mathbf{d}_1)|^2$, a similar lemma as Lemma 3.6 also holds linking $\varepsilon^2(k, \mathbf{d}_1, \mathbf{d}_2)$ and ε^2 . As a consequence, Theorem 3.5 with $d = 2$ can be proved by following the argument of analytic continuation leading to Theorem 2.6. The stability for $d = 3$ follows in a similar manner. Thus, we omit the relevant proofs for brevity.

4. Conclusion

In this paper, we address the inverse random source problems for electromagnetic and biharmonic waves in a unified framework. The source is assumed to be a GMIG random field whose covariance is a classical pseudo-differential operator, and the inverse problem is to determine the principle symbol of the covariance operator for the random source. Uniqueness and stability are achieved by using boundary measurements at multiple wavenumbers. The analysis requires the Dirichlet data only at multiple wavenumbers in a finite interval. The stability estimates consist of the data discrepancy and the logarithmic stability. The result shows that the ill-posedness of the inverse source problems decreases as the wavenumber increases for the data.

A possible continuation of this work is to investigate the inverse random source problems for the stochastic wave equations in inhomogeneous medium. In this case, the approach adopted in this paper is no longer applicable. Another more challenging problem is to study the stability of determining a random potential in wave equations. We hope to report progresses on these problems in the future.

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Appendix A. The GMIG random field

In this section, we introduce the generalized microlocally isotropic Gaussian (GMIG) random fields which are used to characterize the random source considered in this paper. Let $\mathcal{D} := C_0^\infty(\mathbb{R}^d; \mathbb{R})$ and $\mathcal{D}'$ stands for its dual space. Given a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a real-valued random field $g : \Omega \rightarrow \mathcal{D}'$ is called a generalized Gaussian random field if $\omega \mapsto \langle g(\omega), \phi \rangle$ is a real-valued Gaussian random variable for all $\phi \in \mathcal{D}$. Obviously, $g(\eta)$ can be extended to the space of complex-valued smooth functions with compact support as follows

$$\langle g(\omega), \phi \rangle = \langle g(\omega), \Re \phi \rangle + i \langle g(\omega), \Im \phi \rangle, \quad \phi \in C_0^\infty(\mathbb{R}^d; \mathbb{C}).$$

The expectation $\mathbb{E}g \in \mathcal{D}'$ and covariance $\text{Cov}g : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}$ are respectively defined by

$$\begin{aligned} \mathbb{E}g : \phi &\mapsto \mathbb{E}\langle g, \phi \rangle, \quad \phi \in \mathcal{D}, \\ \text{Cov}g : (\phi_1, \phi_2) &\mapsto \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}, \end{aligned}$$

where

$$\text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle) = \mathbb{E}[(\langle g, \phi_1 \rangle - \mathbb{E}\langle g, \phi_1 \rangle)(\langle g, \phi_2 \rangle - \mathbb{E}\langle g, \phi_2 \rangle)]$$

Define the covariance operator C_g by

$$\langle C_g \phi_1, \phi_2 \rangle = \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}.$$

Let $K_g(x, y)$ be the Schwarz kernel of C_g , i.e.,

$$\langle C_g \phi_1, \phi_2 \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_2(y) K_g(x, y) \phi_1(x) dx dy.$$

In this paper, we consider the generalized Gaussian random fields with zero expectation defined as follows.

Definition A.1. A generalized Gaussian random field g with zero expectation is said to be microlocally isotropic of order $-n \in \mathbb{R}$ in a bounded Lipschitz domain $D \subset \mathbb{R}^d$ if its covariance operator C_g is a classical pseudo-differential operator of order $-n$ with the principal symbol $\mu(x)|\xi|^{-n}$, where $\mu(x) \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$ is a real-valued function with $\text{supp } \mu \subset\subset D$. Here μ is called the micro-correlation strength of g .

Let c_g be the symbol of the operator C_g , i.e.,

$$C_g(\phi)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} c_g(x, \xi) \hat{\phi}(\xi) d\xi, \quad \phi \in \mathcal{D},$$

which gives

$$K_g(x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} c_g(x, \xi) d\xi.$$

The following lemma [14, Lemma 2.6] concerns the regularity of the GMIG random field.

Lemma A.2. *Let g be a GMIG random field of order $-n$ in D . Then $g \in W^{\frac{n-d}{2}-\delta, p}(D)$ almost surely for all $\delta > 0$ and $1 < p < \infty$.*

Throughout this paper, we consider GMIG random fields of order $-n$ with $n > d - 1$. In this case, there holds the following results for the kernel [6, Lemma 4.1].

Lemma A.3. *Let g be a GMIG random field of order $-n$ in D . The covariance function $K_g(x, y)$ has the following form:*

(i) *If $a < n - d - 1 \leq a + 1$ with $a = 1, 2, 3, \dots$,*

$$K_g(x, y) = c\mu(x)|x - y|^{n-d} + F_n(x, y),$$

where c is a constant dependent on n and $F_n(x, y) \in C^{a, \gamma}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\gamma \in (0, n - d - a + 1)$.

(ii) *If $d - 1 < n < d$,*

$$K_g(x, y) = c\mu(x)|x - y|^{n-d} + F_n(x, y),$$

where c is a constant dependent on n and $F_m(x, y) \in C^{0, \gamma}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\gamma \in (0, n - d + 1)$.

(iii) *If $n = d$,*

$$K_g(x, y) = c\mu(x) \log |x - y| + F_n(x, y),$$

where c is a constant dependent on $F_n(x, y) \in C^{0, \gamma}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\gamma \in (0, 1)$.

Now we consider generalized Gaussian random vector fields in three dimensions. Its definition is given as follows.

Definition A.4. A vector field $\mathbf{g} = (g_1, g_2, g_3)^\top$ is said to be a real-valued GMIG random field of order $-n$ in D if each component g_j , $j = 1, 2, 3$, is a real-valued GMIG random field of the same order $-n$ in D .

Thus, according to the definition of scalar-valued GMIG, the expectation $\mathbb{E} \mathbf{g} \in (\mathcal{D}')^3$ and covariance $\text{Cov } \mathbf{g} : \mathcal{D}^3 \times \mathcal{D}^3 \rightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned} \mathbb{E} \mathbf{g} : \boldsymbol{\phi} &\mapsto \mathbb{E} \langle \mathbf{g}, \boldsymbol{\phi} \rangle, \quad \boldsymbol{\phi} \in \mathcal{D}^3, \\ \text{Cov } \mathbf{g} : (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) &\mapsto \text{Cov}(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle), \quad \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{D}^3, \end{aligned}$$

where

$$\text{Cov}(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle) = \mathbb{E}[(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle - \mathbb{E} \langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle)(\langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle - \mathbb{E} \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle)].$$

The covariance operator $C_{\mathbf{g}}$ is defined by

$$\langle C_{\mathbf{g}} \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = \text{Cov}(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle), \quad \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{D}^3.$$

Let the 3×3 matrix-valued function $K_{\mathbf{g}}(x, y)$ be the Schwarz kernel of $C_{\mathbf{g}}$, i.e.,

$$\langle C_{\mathbf{g}} \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \boldsymbol{\phi}_2(y)^T K_{\mathbf{g}}(x, y) \boldsymbol{\phi}_1(x) \, dx dy.$$

As $\mathbf{g}$ is microlocally isotropic of order $-n$, the covariance operator $C_{\mathbf{g}}$ is a classical pseudo-differential operator with the principal symbol $\mathbb{M}(x)|\xi|^{-n}$, where $\mathbb{M}(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ is a smooth real-valued matrix function with $\text{supp } \mathbb{M} \subset \subset D$. The matrix $\mathbb{M}$ is called the micro-correlation strength matrix of $\mathbf{g}$.

Appendix B. Proof of Theorem 2.6

We assume $\epsilon < e^{-1}$. Let

$$A = \begin{cases} K^{\frac{2}{3}} E^{\frac{1}{4}}, & 2^{\frac{1}{4}} K^{\frac{1}{3}} < E^{\frac{1}{4}}, \\ K, & E^{\frac{1}{4}} \leq 2^{\frac{1}{4}} K^{\frac{1}{3}}. \end{cases}$$

If $2^{\frac{1}{4}} K^{\frac{1}{3}} < E^{\frac{1}{4}}$, we have $A > 2^{\frac{1}{4}} K > K$. Then for $k \in (K, A]$, applying Lemma 2.12 yields

$$\begin{aligned} \epsilon^2(k, \eta, \mathbf{d}) &\lesssim \epsilon^{2\mu(k)} e^{ck} \\ &\lesssim \exp \left\{ A - 2E((k/K)^4 - 1)^{-\frac{1}{2}} \right\} \\ &\lesssim M^2 \exp \left\{ K^{\frac{2}{3}} E^{\frac{1}{4}} - \frac{2E}{\pi} (k/A)^2 \right\} \\ &\lesssim \exp \left\{ -2K^{\frac{2}{3}} E^{\frac{1}{2}} \left(1 - \frac{1}{2} E^{-\frac{1}{4}} \right) \right\}. \end{aligned}$$

Noticing that $\epsilon < e^{-1}$ implies

$$1 - \frac{1}{2} E^{-\frac{1}{4}} > \frac{1}{2},$$

we have

$$\epsilon^2(k, \eta, \mathbf{d}) \lesssim \exp \left\{ -K^{\frac{2}{3}} E^{\frac{1}{2}} \right\} \lesssim \frac{1}{(K^{\frac{2}{3}} E^{\frac{1}{2}})^N}. \quad (\text{B.1})$$

Here we have used $e^{-x} \leq N!/x^N$ for $N \in \mathbb{N}$ and $x > 0$. Combining (2.29) and (B.1) gives the estimate

$$|\widehat{m}_{jl}(\xi)|^2 \lesssim \frac{1}{A^2} + \frac{1}{(K^{\frac{2}{3}} E^{\frac{1}{2}})^N}, \quad |\xi| \leq A,$$

which yields

$$\int_{|\xi| \leq A^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi \lesssim A^{3\gamma-2} + A^{3\gamma} \frac{1}{(K^{\frac{2}{3}} E^{\frac{1}{2}})^N}. \quad (\text{B.2})$$

By the condition $\|m_{jl}\|_{H^s(\mathbb{R}^3)} \leq Q$, we obtain

$$\int_{|\xi| > A^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi \lesssim \frac{1}{A^{2\gamma s}}. \quad (\text{B.3})$$

Let $N > 1$ and $\gamma = \frac{2}{2s+3}$ such that $\frac{N}{2} - \frac{3}{4}\gamma > \frac{1}{4}(2 - 3\gamma) = \frac{1}{2}\gamma s$. Therefore, in view of (B.2)–(B.3) we have

$$\|m_{jl}\|_{L^2(D)}^2 = \int_{|\xi| \leq A^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi + \int_{|\xi| > A^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi \lesssim E^{-\frac{s}{2s+3}} \left(\frac{1}{K^{\frac{8s}{3(2s+3)}}} + \frac{1}{K^\beta} \right),$$

where $\beta = 1 + \frac{2}{3}N - \frac{12}{3(2s+3)}$. It is easy to verify $\beta > \frac{8s}{3(2s+3)}$ which gives

$$\|m_{jl}\|_{L^2(D)}^2 \lesssim \frac{1}{K^{\frac{8s}{3(2s+3)}} E^{\frac{s}{2s+3}}}. \quad (\text{B.4})$$

If $2^{\frac{1}{4}} K^{\frac{1}{3}} \geq E^{\frac{1}{4}}$, from direct calculations we have

$$\begin{aligned} \|m_{jl}\|_{L^2(D)}^2 &= \int_{|\xi| \leq K^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi + \int_{|\xi| > K^\gamma} |\widehat{m}_{jl}(\xi)|^2 d\xi \\ &\lesssim K^{\frac{6}{3+2s}} \epsilon^2 + \frac{1}{K^{\frac{4s}{2s+3}}} \lesssim K^{\frac{2d}{3+2s}} \epsilon^2 + \frac{1}{K^{\frac{8s}{3(2s+3)}} E^{\frac{s}{2s+3}}}. \end{aligned} \quad (\text{B.5})$$

Combining (B.4) and (B.5), we complete the proof.

Data availability

No data was used for the research described in the article.

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