

## RESEARCH ARTICLE

# Recovering Initial Values and a Random Source Simultaneously for a Damped Wave Equation

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## ABSTRACT

This paper introduces an inverse problem for a stochastic damped wave equation, in which the source is driven by a fractional Brownian motion. The well-posedness of direct problem is obtained by analyzing regularity of the solution for the equivalent stochastic initial value problem in frequency domain. The inverse problem involves recovering two initial values and a random source simultaneously. It is shown that the mean of final observations at two different moments uniquely determines the initial values. Additionally, for inversion of random source, it is demonstrated that its sine modulus can be uniquely determined by the variance of final observations. Based on this sine modulus, recovering the unknown random source can be transformed into a well-known phase retrieval problem, which can be successfully solved using the methodology of phaselift. Finally, numerical examples are presented to demonstrate the effectiveness of this method.

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## 1 | Introduction

Motivated by the significant applications in scientific and industrial fields, there has been remarkable progress in the study of damped wave equations and their variants in the last decade. Several notable contributions have been made in analyzing the regularity and asymptotic behavior of solutions, such as the works by Sjöstrand in [1] on the regularity of solutions. The one-dimensional problem on unbounded domains was investigated in [2], while its three-dimensional counterpart was studied in [3], extending the previous results. The well-posedness of the problem on bounded domains was addressed in [4], and the strongly damped wave equation was considered in [5]. Inverse problems for damped wave equations have also garnered attention, with some recent advancements in inverse source problems and inverse initial value problems, as seen in [6–8]. It should be noted that the terminal value problem for the deterministic strongly damped wave equation poses severe ill-posedness issues, as discussed in [8, 9].

In recent years, there has been an emerging trend of incorporating randomness into inverse source problems to account for uncertainty in real-world scenarios. Notably, research has focused on the Helmholtz equation and diffusion equations, with investigations into the inverse random source problem. In [10], it was demonstrated that the correlation of the random source can be uniquely determined when the correlation of the random wave field is known. Works such as [11–14] have extensively explored the regularity of mild solutions and developed computational methods for recovery, utilizing boundary measurements of the random wave field at multiple frequencies to obtain statistical properties of the random source, such as mean and variance. Furthermore, research has tackled the scenario of incomplete data, as exemplified by [15], which employed Fourier transform and the fundamental solution to analyze mild solutions and proposed a numerical method based on phaselift for recovering unknown random sources. Studies on stochastic wave equations have also examined the asymptotic behavior of solutions, as observed in [16, 17].

Regarding inverse problems with random sources driven by fractional Brownian motion, an inverse source problem has been investigated in [13], while an inverse initial value problem for a damped wave equation with a random source driven by a fractional Brownian motion was studied in [18]. However, although important, research on recovering both initial values and random sources is still in its early stages. A noteworthy global uniqueness result for stochastic wave equations with three unknowns was obtained in [19] using a global Carleman estimate.

Inspired by [13, 15], this paper considers recovery of initial status and a random source simultaneously for a damped wave equation as follows:

$$\begin{cases} \partial_{tt}u(x, t) + a(t)\partial_t u(x, t) - \Delta u(x, t) = f(x)\dot{B}^H(t), & (x, t) \in D \times (0, T], \\ u(x, t) = 0, & (x, t) \in \partial D \times (0, T), \\ u(x, 0) = p(x), \partial_t u(x, 0) = q(x), & x \in \bar{D}, \end{cases} \quad (1.1)$$

where  $D \subset \mathbb{R}^d$  is a bounded domain with Lipschitz boundary  $\partial D$  and  $a(t)$  is a continuous non-negative damping coefficient. Here  $f(x)$ ,  $p(x)$ ,  $q(x)$  are deterministic functions with compact supports in  $D$ .  $B^H(t)$  with Hurst parameter  $H \in (0, 1)$  stands for the temporal fractional Brownian motion (fBm) defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and  $\dot{B}^H(t)$  denotes the formal derivative of  $B^H$  with respect to the time  $t$ . In particular, when  $H = \frac{1}{2}$ , the fBm becomes the classical Brownian motion and  $B^H(t)$  becomes the white noise. More details of the fBm can be found in [13].

In this paper, we are interested in the inverse random source problem, that is, to recover  $f$  and  $p, q$  from the final-time observation, that is,  $u(x, T)$ . For the forward problem (1.1), the uniqueness and existence of path-wise mild solution can be examined in frequency domain by Fourier transform. For the inverse problem, the initial values  $p(x)$ ,  $q(x)$  can be uniquely determined by the mean of  $u(x, T_1)$  and  $u(x, T_2)$ . Moreover, the sine modulus  $|f_k|$  or the Fourier modulus  $|\hat{f}|$  is proved to be uniquely determined by the variance of solution in frequency domain. The recovery of source function  $f$  from  $|\hat{f}|$ , however, does not have the uniqueness since the phase information of  $\hat{f}$  is absence. Inspired by the authors' previous work [15], we adopt the method so-called phaselift which uses random masks to collect additional measurements of the modulus  $|\hat{f}|$  to determine  $|f|$  uniquely.

The paper is organized as follows. Section 2 is devoted to the well-posedness of the forward problem. Section 3 is concerned with the inverse problem, that is, to reconstruct the initial function  $p(x)$ ,  $q(x)$  and source function  $f$  via the mean and variance data from two different moments  $u(x, T_1)$  and  $u(x, T_2)$ . Precisely speaking, the mean is used to recover initial status and the variance data is for random source by solving a phase retrieval problem, respectively. Numerical experiments are implemented in Section 4 to demonstrate the validity of proposed methodology. Section 5 concludes several remarks.

## 2 | Direct Problem and Formulation

In this section, we discuss the well-posedness of the direct problem for Equation (1.1). Let  $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$  be the eigensystem of the operator  $-\Delta$  with the homogeneous Dirichlet boundary condition in  $D$ . It is known that the eigenvalues  $\{\lambda_k\}_{k=1}^{\infty}$  satisfy  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ ,  $\lambda_k \rightarrow \infty$ , as  $k \rightarrow \infty$  and the eigen-functions  $\{\varphi_k\}_{k=1}^{\infty}$  form a complete and orthogonal basis in  $L^2(D)$ . To seek the mild solution of (1.1) in  $L^2(D)$ , we have

$$u(x, t, \omega) = \sum_{k=1}^{\infty} u_k(t, \omega) \varphi_k(x),$$

where  $\omega \in \Omega$  and  $u_k(t, \omega)$  satisfies stochastic ordinary differential equation

$$\begin{cases} \partial_{tt} u_k(t, \omega) + a(t) \partial_t u_k(t, \omega) + \lambda_k u_k(t, \omega) = f_k \dot{B}^H(t), & t \in (0, T), \\ u_k(0) = p_k, \partial_t u_k(0) = q_k. \end{cases} \quad (2.1)$$

Here  $f_k = (f, \varphi_k)_{L^2(D)}$ ,  $p_k = (p, \varphi_k)_{L^2(D)}$  and  $q_k = (q, \varphi_k)_{L^2(D)}$ .

Let  $\vec{u}_k = \begin{bmatrix} u_k \\ \partial_t u_k \end{bmatrix}$ ,  $M_k(t) = \begin{bmatrix} 0 & 1 \\ -\lambda_k & -a(t) \end{bmatrix}$  and  $\vec{f}_k = \begin{bmatrix} 0 \\ f_k \end{bmatrix}$ , then we can rewrite the Equation (2.1) as follows

$$\begin{cases} d\vec{u}_k = M_k(t) \vec{u}_k dt + \vec{f}_k dB^H(t), \\ \vec{u}_k(0) = \begin{bmatrix} p_k \\ q_k \end{bmatrix}. \end{cases} \quad (2.2)$$

Then, the mild solution of the Equation (2.2) takes the form

$$\vec{u}_k(t, \omega) = \Phi_k(t) \left[ \vec{u}_0 + \int_0^t \Phi_k^{-1}(\tau) \vec{f}_k dB^H(\tau) \right], \quad (2.3)$$

where  $\Phi_k(t)$  is the fundamental matrix which satisfies the initial problem  $\Phi_k'(t) = M_k(t) \Phi_k(t)$ ,  $\Phi_k(0) = I_{2 \times 2}$ . Here  $I_{2 \times 2}$  denotes an identity matrix of second order. Denote  $\Phi_k(t) = \begin{bmatrix} \phi_{k,1}(t) & \phi_{k,2}(t) \\ \phi'_{k,1}(t) & \phi'_{k,2}(t) \end{bmatrix}$  and  $\det \Phi_k(t)$  as the determinant of  $\Phi_k(t)$ . Notice

$$\begin{cases} [\det \Phi_k(t)]' = \text{trace}(M_k(t)) \det \Phi_k(t), \\ \det \Phi_k(0) = 1. \end{cases}$$

Hence  $\det \Phi_k(t) = e^{-\int_0^t a(s) ds} > 0$ ,  $k \in \mathbb{N}$  and  $\Phi_k(t)^{-1} = \frac{1}{\det \Phi_k(t)} \begin{bmatrix} \phi'_{k,2}(t) & -\phi_{k,2}(t) \\ -\phi'_{k,1}(t) & \phi_{k,1}(t) \end{bmatrix}$ . Substituting it to the mild solution (2.3), we get

$$\begin{aligned} u_k(t, \omega) &= \phi_{k,1}(t) p_k + \phi_{k,2}(t) q_k \\ &+ f_k \int_0^t [-\phi_{k,1}(t) \phi_{k,2}(\tau) + \phi_{k,2}(t) \phi_{k,1}(\tau)] e^{\int_0^\tau a(s) ds} dB^H(\tau). \end{aligned} \quad (2.4)$$

Denote  $g_k(t, \tau) = [-\phi_{k,1}(t) \phi_{k,2}(\tau) + \phi_{k,2}(t) \phi_{k,1}(\tau)] e^{\int_0^\tau a(s) ds} \chi_{\{\tau \leq t\}}$ , where  $t \in (0, T)$ . Then  $u_k(t, \omega)$  can be rewritten as:

$$u_k(t, \omega) = I_{k,1}(t) + I_{k,2}(t) + I_{k,3}(t, \omega), \quad (2.5)$$

where  $I_{k,1}(t) = \phi_{k,1}(t) p_k$ ,  $I_{k,2}(t) = \phi_{k,2}(t) q_k$ ,  $I_{k,3}(t, \omega) = f_k \int_0^T g_k(t, \tau) dB^H(\tau)$ . Then the stochastic process  $u : D \times [0, T] \rightarrow L^2(D)$  defined by

$$u(x, t, \omega) = \sum_{k=1}^{\infty} (I_{k,1}(t) + I_{k,2}(t) + I_{k,3}(t, \omega)) \varphi_k(x), \quad (2.6)$$

is a mild solution to the initial-boundary value problem (1.1). In the following, we show this mild solution (2.6) is well-defined.

**Lemma 1.** Let  $\Phi_k'(t) = M_k(t) \Phi_k(t)$ ,  $\Phi_k(0) = I_{2 \times 2}$ . Then

$$\begin{cases} |\phi_{k,1}(t)| \leq 1, |\phi_{k,2}(t)| \leq \frac{1}{\sqrt{\lambda_k}}, & \forall t \in [0, T], \\ \|g_k(t, \tau)\|_{L^2((0,T) \times (0,T))}^2 \lesssim \frac{T^2 e^{2\|a\|_{L^1(0,T)}}}{\lambda_k}. \end{cases} \quad (2.7)$$

Hereafter, the notation  $a \lesssim b$  stand for  $a \leq Cb$  where  $C$  is a positive constant whose value is not required but should be clear from the context.

**Proof.** Since both  $\phi_{k,1}$  and  $\phi_{k,2}$  in the fundamental matrix  $\Phi_k$  satisfy  $\phi_k'' + a(t)\phi_k' + \lambda_k\phi_k = 0$ , multiplying  $\phi_k'(t)$  on both sides and integration by parts gives

$$\frac{1}{2}\phi_k'^2(\tau)|_0^t + \int_0^t a(\tau)\phi_k'(\tau)d\tau + \frac{1}{2}\lambda_k\phi_k^2(\tau)|_0^t = 0.$$

Combining the initial conditions and  $a(t) > 0$ , we get  $\phi_{k,1}$  and  $\phi_{k,2}$  respectively:

$$\begin{cases} \frac{1}{2}\lambda_k\phi_{k,1}^2(t) - \frac{1}{2}\lambda_k = -\int_0^t a(\tau)\phi_{k,1}'(\tau)d\tau - \frac{1}{2}\phi_{k,1}'^2(t) \leq 0, \\ -\frac{1}{2} + \frac{1}{2}\lambda_k\phi_{k,2}^2(t) = -\int_0^t a(\tau)\phi_{k,2}'(\tau)d\tau - \frac{1}{2}\phi_{k,2}'^2(t) \leq 0. \end{cases}$$

Then we immediately obtain

$$|\phi_{k,1}(t)| \leq 1, \quad |\phi_{k,2}(t)| \leq \frac{1}{\sqrt{\lambda_k}}.$$

Moreover,

$$\begin{aligned} & \|g_k(t, \tau)\|_{L^2((0,T) \times (0,T))}^2 \\ &= \int_0^T \int_0^T [-\phi_{k,1}(t)\phi_{k,2}(\tau) + \phi_{k,2}(t)\phi_{k,1}(\tau)]^2 e^{2\int_0^\tau a(s)ds} \chi_{\{\tau \leq t\}} d\tau dt \\ &\leq \int_0^T \int_0^T [|\phi_{k,1}(t)\phi_{k,2}(\tau)| + |\phi_{k,2}(t)\phi_{k,1}(\tau)|]^2 e^{2\|a\|_{L^1}} d\tau dt \\ &\leq \int_0^T \int_0^T \left(\frac{2}{\sqrt{\lambda_k}}\right)^2 e^{2\|a\|_{L^1}} d\tau dt \leq T^2 \left(\frac{2}{\sqrt{\lambda_k}}\right)^2 e^{2\|a\|_{L^1}} \leq \frac{4T^2 e^{2\|a\|_{L^1}}}{\lambda_k}, \end{aligned} \quad (2.8)$$

which completes the proof of (2.7).  $\square$

**Lemma 2.** Assume  $f(x), p(x), q(x) \in L^2(D)$  and non-negative function  $a(t) \in C(0, T)$ . Then the stochastic differential Equation (2.1) admits a unique mild solution given by (2.4). Moreover, the solution  $u_k(t)$  satisfy the estimates

$$\begin{cases} \|I_{k,1}(\cdot)\|_{L^2(0,T)}^2 \lesssim T|p_k|^2, \\ \|I_{k,2}(\cdot)\|_{L^2(0,T)}^2 \lesssim T|q_k|^2, \\ \mathbb{E}[\|I_{k,3}(\cdot, \omega)\|_{L^2(0,T)}^2] \lesssim T^{2H+1} e^{2\|a\|_{L^1(0,T)}} |f_k|^2. \end{cases} \quad (2.9)$$

**Proof.** The existence and uniqueness of the mild solution is obvious hence the proof is omitted. Here, we only show the proof of (2.9). According to the definition of  $I_{k,1}$ ,  $I_{k,2}$  and Lemma 1, it immediately gives

$$\begin{cases} \|I_{k,1}(\cdot)\|_{L^2(0,T)}^2 = \int_0^T |\phi_{k,1}(t)p_k|^2 dt \leq \int_0^T |p_k|^2 dt = T|p_k|^2, \\ \|I_{k,2}(\cdot)\|_{L^2(0,T)}^2 = \int_0^T |\phi_{k,2}(t)q_k|^2 dt \leq \int_0^T \frac{|q_k|^2}{\lambda_k} dt = \frac{T|q_k|^2}{\lambda_k} \leq \frac{T|q_k|^2}{\lambda_1}. \end{cases} \quad (2.10)$$

Next we consider the estimate of  $I_{k,3}$  for different  $H$ . For the case  $H = \frac{1}{2}$ , it follows from Ito's isometry that

$$\begin{aligned} \mathbb{E}[\|I_{k,3}(\cdot, \omega)\|_{L^2(0,T)}^2] &= \mathbb{E}\left[\int_0^T \left|f_k \int_0^t g_k(t, \tau) dB^{\frac{1}{2}}(\tau)\right|^2 dt\right] \\ &= \int_0^T \mathbb{E}\left[\int_0^T f_k g_k(t, \tau) dB^{\frac{1}{2}}(\tau)\right]^2 dt = \int_0^T \int_0^T |f_k g_k(t, \tau)|^2 d\tau dt \\ &= |f_k|^2 \|g_k(t, \tau)\|_{L^2((0,T) \times (0,T))}^2 \lesssim \frac{|f_k|^2 T^2 e^{2\|a\|_{L^1(0,T)}}}{\lambda_k} \leq \frac{T^2}{\lambda_1} |f_k|^2 e^{2\|a\|_{L^1(0,T)}}. \end{aligned} \quad (2.11)$$

For the case  $H \in (0, \frac{1}{2})$ , we have from [14] that

$$\begin{aligned} \mathbb{E} \left[ \|I_{k,3}(\cdot, \omega)\|_{L^2(0,T)}^2 \right] &= \mathbb{E} \left[ \int_0^T |f_k \int_0^T g_k(t, \tau) dB^H(\tau)|^2 dt \right] \\ &= \int_0^T \mathbb{E} \left[ \int_0^T f_k g_k(t, \tau) dB^H(\tau) \right]^2 dt \\ &= |f_k|^2 \int_0^T \int_0^T \left\{ c_H \left[ \left( \frac{T}{\tau} \right)^{H-\frac{1}{2}} (T-\tau)^{H-\frac{1}{2}} - (H-\frac{1}{2}) \tau^{\frac{1}{2}-H} \int_\tau^T u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du \right] g_k(t, \tau) \right. \\ &\quad \left. + \int_\tau^T (g_k(t, u) - g_k(t, \tau)) c_H \left( \frac{u}{\tau} \right)^{H-\frac{1}{2}} (H-\frac{1}{2}) (u-\tau)^{H-\frac{3}{2}} du \right\}^2 d\tau dt \\ &\lesssim |f_k|^2 \int_0^T \left\{ \int_0^T \left[ \left( \frac{T}{\tau} \right)^{H-\frac{1}{2}} (T-\tau)^{H-\frac{1}{2}} g_k(t, \tau) \right]^2 d\tau \right. \\ &\quad \left. + \int_0^T \left[ \tau^{\frac{1}{2}-H} \left( \int_\tau^T u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du \right) g_k(t, \tau) \right]^2 d\tau \right. \\ &\quad \left. + \int_0^T \left[ \int_\tau^T (g_k(t, u) - g_k(t, \tau)) \left( \frac{u}{\tau} \right)^{H-\frac{1}{2}} (u-\tau)^{H-\frac{3}{2}} du \right]^2 d\tau \right\} dt \\ &=: |f_k|^2 (S_1 + S_2 + S_3), \end{aligned}$$

where  $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}}$  and  $\beta(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$ . Then simple calculation gives

$$\begin{aligned} S_1 &= \int_0^T \int_0^T \left( \frac{T}{\tau} \right)^{2H-1} (T-\tau)^{2H-1} g_k^2(t, \tau) d\tau dt \\ &\leq \int_0^T \int_0^T \left( \frac{T}{\tau} \right)^{2H-1} (T-\tau)^{2H-1} \frac{4}{\lambda_k} e^{2\|a\|_{L^1}} d\tau dt \\ &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T (T-\tau)^{2H-1} d\tau \leq \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} T^{2H} \lesssim \frac{T^{2H+1}}{\lambda_1} e^{2\|a\|_{L^1}}. \end{aligned} \quad (2.12)$$

Similarly, we have

$$\begin{aligned} S_2 &= \int_0^T \int_0^T \tau^{1-2H} \left( \int_\tau^T u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du \right)^2 g_k^2(t, \tau) d\tau dt \\ &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T \tau^{1-2H} \left( \int_\tau^T u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du \right)^2 d\tau. \end{aligned} \quad (2.13)$$

Using the binomial expansion leads to

$$\begin{aligned} \int_\tau^T u^{H-\frac{3}{2}} (u-\tau)^{H-\frac{1}{2}} du &= \int_\tau^T u^{2H-2} \left[ \sum_{n=0}^{\infty} \binom{H-\frac{1}{2}}{n} \left( -\frac{\tau}{u} \right)^n \right] du \\ &= \sum_{n=0}^{\infty} \binom{H-\frac{1}{2}}{n} (-1)^n \tau^n \left( \frac{T^{2H-1-n} - \tau^{2H-1-n}}{2H-1-n} \right) \\ &\leq (T^{2H-1} - \tau^{2H-1}) \sum_{n=0}^{\infty} \binom{H-\frac{1}{2}}{n} \frac{(-1)^n}{2H-1-n} \lesssim T^{2H-1} - \tau^{2H-1}. \end{aligned} \quad (2.14)$$

Combining the above two estimates in (2.13) and (2.14), we obtain that

$$S_2 \lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T \tau^{1-2H} (T^{2H-1} - \tau^{2H-1})^2 d\tau$$

$$\begin{aligned} &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T \tau^{1-2H} (T^{4H-2} + \tau^{4H-2}) d\tau \\ &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T \tau^{2H-1} d\tau \lesssim \frac{T^{2H+1}}{\lambda_1} e^{2\|a\|_{L^1}}. \end{aligned}$$

By the mean value theorem, there holds

$$\begin{aligned} S_3 &= \int_0^T \int_0^T \left[ \int_\tau^T (g_k(t, u) - g_k(t, \tau)) \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u - \tau)^{H-\frac{3}{2}} du \right]^2 d\tau dt \\ &\lesssim \int_0^T \int_0^T \left[ \int_\tau^T \frac{1}{\sqrt{\lambda_k}} |e^{\|a\|_{L^1}}| \left(\frac{u}{\tau}\right)^{H-\frac{1}{2}} (u - \tau)^{H-\frac{3}{2}} du \right]^2 d\tau dt \\ &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T \left[ \int_\tau^T (u - \tau)^{H-\frac{3}{2}} du \right]^2 d\tau \\ &\lesssim \frac{T}{\lambda_k} e^{2\|a\|_{L^1}} \int_0^T (T - \tau)^{2H-1} d\tau \lesssim \frac{T^{2H+1}}{\lambda_1} e^{2\|a\|_{L^1}}. \end{aligned} \quad (2.15)$$

Combining (2.12), (2.13) and (2.15), we obtain for  $H \in (0, \frac{1}{2})$  that

$$\mathbb{E} \left[ \|I_{k,3}(\cdot, \omega)\|_{L^2(0,T)}^2 \right] \lesssim \frac{T^{2H+1}}{\lambda_1} |f_k|^2 e^{2\|a\|_{L^1}}. \quad (2.16)$$

For the case  $H \in (\frac{1}{2}, 1)$ , we also have from [14] and Lemma 1 that

$$\begin{aligned} \mathbb{E} \left[ \|I_{k,3}(\cdot, \omega)\|_{L^2(0,T)}^2 \right] &= \mathbb{E} \left[ \int_0^T |f_k| \int_0^T g_k(t, \tau) dB^H(\tau) \right]^2 dt \\ &= \alpha_H |f_k|^2 \int_0^T \left[ \int_0^T \int_0^T g_k(t, \tau_1) g_k(t, \tau_2) |\tau_1 - \tau_2|^{2H-2} d\tau_1 d\tau_2 \right] dt \\ &\leq \alpha_H |f_k|^2 \int_0^T \left[ \int_0^T \int_0^T \frac{4}{\lambda_k} e^{2\|a\|_{L^1}} |\tau_1 - \tau_2|^{2H-2} d\tau_1 d\tau_2 \right] dt \\ &\leq \frac{4\alpha_H T}{\lambda_k} |f_k|^2 e^{2\|a\|_{L^1}} \left[ \int_0^T \int_0^T |\tau_1 - \tau_2|^{2H-2} d\tau_1 d\tau_2 \right] \\ &\lesssim \frac{T^{2H+1}}{\lambda_1} |f_k|^2 e^{2\|a\|_{L^1}}, \end{aligned} \quad (2.17)$$

with  $\alpha_H = H(2H - 1)$ . Thus completes the proof from (2.11), (2.16) and (2.17).  $\square$

**Theorem 1.** Assume  $f(x), p(x), q(x) \in L^2(D)$  and non-negative function  $a(t) \in C(0, T)$ . Then the initial boundary value problem of the stochastic differential Equation (1.1) admits a unique solution  $u$  satisfying

$$\mathbb{E} \|u\|_{L^2(D \times (0,T))}^2 \lesssim T \|p\|_{L^2(D)}^2 + T \|q\|_{L^2(D)}^2 + T^{2H+1} e^{2\|a\|_{L^1(0,T)}} \|f\|_{L^2(D)}^2. \quad (2.18)$$

*Proof.* The existence and uniqueness of the mild solution (2.6) can be easily obtained since Theorem 2 indicates the existence and uniqueness of the solution of Equation (2.1) which is equivalent to Equation (1.1). Note that the mild solution (2.6) satisfies

$$\begin{aligned} \|u(\cdot, t)\|_{L^2(D)}^2 &= \left\| \sum_{k=1}^{\infty} (I_{k,1}(t) + I_{k,2}(t) + I_{k,3}(t, \omega)) \varphi_k(\cdot) \right\|_{L^2(D)}^2 \\ &= \sum_{k=1}^{\infty} (I_{k,1}(t) + I_{k,2}(t) + I_{k,3}(t, \omega))^2 \\ &\leq 3 \sum_{k=1}^{\infty} (I_{k,1}^2(t) + I_{k,2}^2(t) + I_{k,3}^2(t, \omega)). \end{aligned}$$

Hence, applying Lemma 2 and Parseval's identity we have

$$\begin{aligned}
 \mathbb{E} \left[ \|u\|_{L^2(D \times (0,T))}^2 \right] &= \mathbb{E} \left[ \int_0^T \|u(\cdot, t)\|_{L^2(D)}^2 dt \right] \\
 &\lesssim \mathbb{E} \left[ \int_0^T \left( \sum_{k=1}^{\infty} I_{k,1}^2(t) + \sum_{k=1}^{\infty} I_{k,2}^2(t) + \sum_{k=1}^{\infty} I_{k,3}^2(t, \omega) \right) dt \right] \\
 &= \int_0^T \left( \sum_{k=1}^{\infty} I_{k,1}^2(t) \right) dt + \int_0^T \left( \sum_{k=1}^{\infty} I_{k,2}^2(t) \right) dt + \mathbb{E} \left[ \int_0^T \left( \sum_{k=1}^{\infty} I_{k,3}^2(t, \omega) \right) dt \right] \\
 &= \sum_{k=1}^{\infty} \|I_{k,1}\|_{L^2(0,T)}^2 + \sum_{k=1}^{\infty} \|I_{k,2}\|_{L^2(0,T)}^2 + \sum_{k=1}^{\infty} \mathbb{E} \left[ \|I_{k,3}\|_{L^2(0,T)}^2 \right] \\
 &\lesssim \sum_{k=1}^{\infty} T |p_k|^2 + \sum_{k=1}^{\infty} T |q_k|^2 + \sum_{k=1}^{\infty} T^{2H+1} e^{2\|a\|_{L^1(0,T)}} |f_k|^2 \\
 &= T \|p\|_{L^2(D)}^2 + T \|q\|_{L^2(D)}^2 + T^{2H+1} e^{2\|a\|_{L^1(0,T)}} \|f\|_{L^2(D)}^2,
 \end{aligned}$$

which completes the proof.  $\square$

### 3 | The Inverse Problem

This section investigates the inverse problem of recovering the initial conditions  $p, q$  and the random source  $f$  from the mean and variance of the measured wave field.

#### 3.1 | Reconstruction of Initial Values

To reconstruct initial values  $p$  and  $q$ , we consider the reconstruction of  $p_k$  and  $q_k$  first. Then the uniqueness and instability will be discussed. It follows from (2.4) that the mean of the solution at two different moments, that is,  $t = T_1$  and  $t = T_2$ , satisfies

$$\begin{cases} \mathbb{E}[u_k(T_1)] = \phi_{k,1}(T_1)p_k + \phi_{k,2}(T_1)q_k, \\ \mathbb{E}[u_k(T_2)] = \phi_{k,1}(T_2)p_k + \phi_{k,2}(T_2)q_k. \end{cases} \quad (3.1)$$

**Lemma 3.** For equation

$$\phi''(t) + a(t)\phi'(t) + \lambda\phi(t) = 0, \quad t \in (0, T), \quad (3.2)$$

where  $a(t) \in C(0, T)$  and  $\lambda$  is a constant. Suppose  $\phi = \phi_0(t)$  is a non-trivial solution of (3.2) on  $(0, T)$ , then it holds that:

1.  $\phi_0(t)$  only has simple zero points on  $(0, T)$ . Moreover,  $\phi_0(t)$  has most finite zeros on any finite closed interval, so each zero point is isolated.
2. The solution  $\phi(t)$  of (3.2) takes the following form:

$$\phi(t) = \phi_0(t) \left[ C_1 + C_2 \int_0^t \frac{1}{\phi_0^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau \right], \quad (3.3)$$

where  $C_1$  and  $C_2$  are constants.

*Proof.* Suppose there exists  $s_0 \in (0, T)$  which is a non simple zero of  $\phi_0(t)$ , then  $\phi_0(s_0) = \phi_0'(s_0) = 0$ . Substituting it to Equation (3.2) gives  $\phi_0''(s_0) = 0$ , and hence  $\phi_0(t) \equiv 0, \forall t \in (0, T)$ , which leads to a contradiction.

Moreover, suppose  $\phi_0(t)$  has infinite zeros on a finite closed interval. Then there exists an accumulation point  $t_0 \in (0, T)$  and a series of zero points  $\{t_i\}_{i=1}^{\infty}$  satisfying  $\lim_{i \rightarrow \infty} t_i = t_0$  and  $\phi_0(t_i) = 0, i = 1, 2, \dots, \infty$ . According to the continuity,

we have

$$\phi_0(t_0) = \lim_{i \rightarrow \infty} \phi_0(t_i) = 0, \text{ and } \phi'_0(t_0) = \lim_{i \rightarrow \infty} \frac{\phi_0(t_i) - \phi_0(t_0)}{t_i - t_0} = 0.$$

Then  $t_0$  is a non simple zero point, which leads to a contradiction. Thus, we complete the proof of (1).

Next we proceed to proving (2). Using the zero points of  $\phi_0$  to divide the interval  $(0, T)$  into the union of open intervals, which guarantees that  $\phi_0$  is non-vanishing on each open interval. Suppose  $\phi = \phi(t)$  is a solution of Equation (3.2), then  $\begin{vmatrix} \phi_0(t) & \phi(t) \\ \phi'_0(t) & \phi'(t) \end{vmatrix} = \phi_0 \phi' - \phi'_0 \phi = C e^{-\int_0^t a(s) ds}$ . Multiplying  $\frac{1}{\phi_0^2}$  on both sides, we have  $\frac{d}{dt} \left( \frac{\phi}{\phi_0} \right) = \frac{C}{\phi_0^2} e^{-\int_0^t a(s) ds}$  on each open interval divided by zero points. After integration, we derive  $\phi(t) = \phi_0(t) \left[ C_1 + C_2 \int_0^t \frac{1}{\phi_0^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau \right]$ , where  $t$  satisfies  $\phi_0(t) \neq 0$ . Next, we consider the limits of solution on both sides at each endpoint of each interval. Suppose  $\zeta \in (0, T)$  is a zero point of  $\phi_0$ , then  $\phi_0(\zeta) = 0 \neq \phi'_0(\zeta)$ . Hence, utilizing L'Hospital law yields

$$\lim_{t \rightarrow \zeta} \phi_0(t) \int_0^t \frac{1}{\phi_0^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau = \lim_{t \rightarrow \zeta} \left[ \frac{\frac{1}{\phi_0^2(t)} e^{-\int_0^t a(\eta) d\eta}}{-\frac{\phi'_0(t)}{\phi_0^2(t)}} \right] = -\frac{1}{\phi'_0(\zeta)} e^{-\int_0^\zeta a(\eta) d\eta}.$$

Then the continuity and uniqueness of  $\phi$  gives rise to

$$\phi(\zeta) = \lim_{t \rightarrow \zeta} \phi_0(t) \left[ C_1 + C_2 \int_0^t \frac{1}{\phi_0^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau \right].$$

Thus completes the proof of (2).  $\square$

**Theorem 2.** Assume  $p(x), q(x) \in L^2(D)$  and  $a(t) \in C(0, T)$ . Then  $p_k$  and  $q_k$  can be uniquely determined by the data  $\mathbb{E}[u_k(T_1)]$  and  $\mathbb{E}[u_k(T_2)]$ ,  $k \in \mathbb{N}$ , where  $T_1, T_2$  can be selected arbitrarily in  $(0, T)$  except for a set of at most countable discrete points.

*Proof.* Without losing generality, we take  $T_1, T_2$  satisfying  $\phi_{k,1}(T_1) \neq 0$  and  $\phi_{k,1}(T_2) \neq 0$ . Then it follows from (3.1) that the coefficient determinant of the linear system (3.1) can be written as  $\phi_{k,1}(T_1)\phi_{k,2}(T_2) - \phi_{k,2}(T_1)\phi_{k,1}(T_2)$ . By Lemma 3, we have

$$\phi_{k,2}(t) = \phi_{k,1}(t) \int_0^t \frac{1}{\phi_{k,1}^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau. \quad (3.4)$$

Hence the coefficient determinant

$$\begin{aligned} & \phi_{k,1}(T_1)\phi_{k,2}(T_2) - \phi_{k,2}(T_1)\phi_{k,1}(T_2) \\ &= \phi_{k,1}(T_1)\phi_{k,1}(T_2) \int_{T_1}^{T_2} \frac{1}{\phi_{k,1}^2(\tau)} e^{-\int_0^\tau a(\eta) d\eta} d\tau \neq 0. \end{aligned}$$

Therefore,  $p_k$  and  $q_k$  can be uniquely determined by the data  $\mathbb{E}[u_k(T_1)]$  and  $\mathbb{E}[u_k(T_2)]$ ,  $k \in \mathbb{N}$  from (3.1).  $\square$

By Theorem 2, the inverse initial value problem admits a unique solution and we can reconstruct the initial status  $p(x), q(x)$  using  $p_k, q_k, k \in \mathbb{N}$ . However, it lacks stability in actual computations for the condition number of coefficient matrix of (3.1) goes to infinity as  $k \rightarrow \infty$ , which is indicated in following theorem.

**Theorem 3.** For any fixed  $k \in \mathbb{N}$ , the coefficient determinant of Equation (3.1) holds the estimate that

$$|\phi_{k,1}(T_1)\phi_{k,2}(T_2) - \phi_{k,2}(T_1)\phi_{k,1}(T_2)| \lesssim \frac{1}{\sqrt{\lambda_k}}.$$

*Proof.* It follows immediately from Lemma 1 that

$$|\phi_{k,1}(T_1)\phi_{k,2}(T_2) - \phi_{k,2}(T_1)\phi_{k,1}(T_2)| \leq \frac{1}{\sqrt{\lambda_k}} + \frac{1}{\sqrt{\lambda_k}} \leq \frac{2}{\sqrt{\lambda_k}}.$$

$\square$

### 3.2 | Reconstruction of Random Source via Phase Retrieval

To reconstruct the source function  $f(x)$ , we develop a two-stage method. The first is to recover the modulus of  $f_k$  explicitly from the variance of measured data. Then apply a phase retrieval method to reconstruct  $|f|$  from phaseless data  $|f_k|$ . It is clear that a stable inversion for the random source can not be guaranteed as the coherent ill-posedness arising from inverse problems, in particular for general  $H \in (0, 1)$ . Below, we only consider the case of  $H = \frac{1}{2}$ , where the uniqueness can be obtained.

First we show a uniqueness of determining  $|f_k|$ . By Ito's isometry, it follows from (2.4) that the mean and variance of the solution at  $t = T$  satisfies

$$\mathbb{E}[u_k(T)] = \phi_{k,1}(T)p_k + \phi_{k,2}(T)q_k, \quad (3.5)$$

and

$$\mathbb{V}[u_k(T)] = |f_k|^2 \int_0^T |g_k(T, \tau)|^2 d\tau. \quad (3.6)$$

where  $g_k(T, \tau) = [-\phi_{k,1}(T)\phi_{k,2}(\tau) + \phi_{k,2}(T)\phi_{k,1}(\tau)]e^{\int_0^\tau a(s)ds} \chi_{\{\tau \leq T\}}$  and  $\phi_1, \phi_2$  are given in fundamental matrix  $\Phi_k$ .

**Lemma 4.** For any  $k \in \mathbb{N}$ ,  $0 < \int_0^T |g_k(T, \tau)|^2 d\tau \lesssim \frac{1}{\lambda_k}$ .

*Proof.* Suppose  $\int_0^T |g_k(T, \tau)|^2 d\tau = 0$ . Notice that  $\phi_1, \phi_2$  are given by fundamental matrix  $\Phi_k$  of the non-autonomous system for the ordinary differential equation:  $\Phi_k'(t) = M_k(t)\Phi_k(t)$ ,  $\Phi_k(0) = I$ . If  $\int_0^T |g_k(T, \tau)|^2 d\tau = 0$  holds, then

$$-\phi_{k,1}(T)\phi_{k,2}(\tau) + \phi_{k,2}(T)\phi_{k,1}(\tau) = 0, \quad a.e. \quad \tau \in [0, T].$$

Since  $\det \Phi_k(t) \neq 0$  for  $t \in [0, T]$ , we derive that  $\phi_1^2(T) + \phi_2^2(T) \neq 0$  and  $\phi_1(\tau), \phi_2(\tau)$  are linear independent, which means  $-\phi_1(T)\phi_2(\tau) + \phi_2(T)\phi_1(\tau) \neq 0$ . Thus leads to a contradiction.

Next we show the latter inequality by direct computations as Lemma 1.

$$\begin{aligned} \int_0^T |g_k(T, \tau)|^2 d\tau &= \int_0^T [-\phi_{k,1}(T)\phi_{k,2}(\tau) + \phi_{k,2}(T)\phi_{k,1}(\tau)]^2 e^{2\int_0^\tau a(s)ds} \chi_{\{\tau \leq T\}} d\tau \\ &\leq \int_0^T [|\phi_{k,1}(T)\phi_{k,2}(\tau)| + |\phi_{k,2}(T)\phi_{k,1}(\tau)|]^2 e^{2\|a\|_{L^1}} d\tau \\ &\leq \int_0^T \left( \frac{2}{\sqrt{\lambda_k}} \right)^2 e^{2\|a\|_{L^1}} d\tau \leq \frac{4T}{\lambda_k} e^{2\|a\|_{L^1}}. \end{aligned}$$

□

**Theorem 4.** Assume  $f(x) \in L^2(D)$ . Then the modulus of  $f_k = (f, \varphi_k)_{L^2(0,T)}$  can be uniquely determined by the data  $\mathbb{V}[u_k(T)]$ ,  $k \in \mathbb{N}$ .

*Remark 1.* The result can be immediately derived from (3.6) and Lemma 4. Moreover, it is obvious that the reconstruction of  $|f_k|$  admits a unique solution but it lacks stability in actual computations for the denominator  $\int_0^T |g_k(T, \tau)|^2 d\tau$  goes to zero as  $k \rightarrow \infty$ .

Once  $|f_k|$  is obtained, we can proceed to the second stage, that is, recovering  $|f|$  via phase retrieval. For simplicity, we take  $D = (0, L)$ . It should be noted that the following method can also be extended to multidimensional case. Take the eigen-functions  $\{\varphi_k\}_{k=1}^\infty$  to be  $\left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \right\}_{k=1}^\infty$ , and the corresponding  $\lambda_k = \left(\frac{k\pi}{L}\right)^2$ ,  $k \in \mathbb{N}$ . Accordingly,  $f_k$  can be viewed as a discrete sine transform (DST) of  $f$ . Noting the connection between discrete sine transform and discrete Fourier transform (DFT), it is easy to know that the DST data of  $f$  is equivalent to DFT data of  $f$ . Hence, we only need to consider recovering  $|f|$  from the modulus of DFT data of  $f$ , which can be characterized as a well-known phase retrieval problem and solved by method of phaselift in [20].

Suppose  $\mathbf{f} = (f(x_1), \dots, f(x_N))^T = (f^1, \dots, f^N) \in \mathbb{C}^N$  to be the discrete unknowns of  $f$ , and  $\hat{\mathbf{f}} = (f_1, \dots, f_N)^T \in \mathbb{C}^N$  be its  $N$ -point discrete Fourier transform. Denote by

$$\mathbf{h}^{(j)} = \left( h_1^{(j)}, \dots, h_N^{(j)} \right)^T := \left( 1, e^{i \frac{2\pi(j-1)}{N}}, \dots, e^{i \frac{2\pi(j-1)(N-1)}{N}} \right)^T. \quad (3.7)$$

Then  $f_j = \langle \mathbf{h}^{(j)}, \mathbf{f} \rangle = \sum_{n=1}^N h_n^{(j)} f^n$ . Hence the discrete phase retrieval problem can be formulated as follows:

$$\begin{aligned} & \text{find } \mathbf{f} \\ & \text{subject to } |f_j|^2 := \left| \langle \mathbf{h}^{(j)}, \mathbf{f} \rangle \right|^2, \quad j = 1, \dots, N. \end{aligned} \quad (3.8)$$

Generally speaking, there are mainly two ways to solve phase retrieval problems. One is developing a modified model with a prior information, and the other is taking more measurements. These two methods can be implemented in several ways depending on the realistic demands. For instance, assuming some *a priori* information for the signal aims to reduce the number of unknowns, such as the support constraints [21], positivity or real-valuedness [22]. Alternatively, taking more measurements, which intends to increase the number of equations instead, can be done by using of masks [23], optical gratings [24], oblique illuminations [25], etc. Among these methods, the phaselift is an efficient and rigorous numerical method to solve the phase retrieval problem (3.8) based on the semi-definite programming method. This approach consists of two folds: multiple structured illuminations and formulation of the phase recovery as a matrix completion problem. More details may be found in [20].

First we introduce the structured illumination. Suppose the illumination schemes collecting the diffraction pattern of the modulated object  $\{w_n f^n\}_{n=1, \dots, N}$  are available, where the patterns or waveforms  $\mathbf{w} = (w_1, \dots, w_N)^T \in \mathbb{C}^N$  may be selected in advance by masking, optical grating or ptychography. In this paper, we will utilize two different masks to derive more measurement in numerical examples.

Next we discuss the formulation of the phase recovery as a matrix completion problem. The phaselift method suggests lifting up the problem of recovering a vector from quadratic constraints into a recovering a rank-one matrix from affine constraints, and relaxing the combinatorial problem into a convenient convex program. Suppose the quadratic measurements are of following the form:

$$\tilde{\mathbf{f}} = (\tilde{f}_1, \dots, \tilde{f}_N)^T := \left( \left| \langle \mathbf{a}^{(1)}, \mathbf{f} \rangle \right|^2, \dots, \left| \langle \mathbf{a}^{(N)}, \mathbf{f} \rangle \right|^2 \right)^T,$$

where  $\mathbf{a}^{(j)}$  can be expressed by the diffraction pattern  $\mathbf{w}$  in the form  $\mathbf{a}^{(j)} = \left( h_1^{(j)} w_1, \dots, h_N^{(j)} w_N \right)^T$ . Then the phase retrieval problem can be transformed to the feasibility problem:

$$\begin{aligned} & \text{find } \mathbf{f} \\ & \text{subject to } \tilde{f}_j := \left| \langle \mathbf{a}^{(j)}, \mathbf{f} \rangle \right|^2, \quad j = 1, \dots, N. \end{aligned} \quad (3.9)$$

As mentioned earlier, quadratic measurements can be lifted up and interpreted as linear measurements about rank-one matrix  $\mathbf{F} = \mathbf{f} \mathbf{f}^*$ . Denote  $\mathbf{A}^{(j)}$  a ranked-one matrix by  $\mathbf{A}^{(j)} := \mathbf{a}^{(j)} (\mathbf{a}^{(j)})^*$ . Then we have

$$\tilde{f}_j = \left| \langle \mathbf{a}^{(j)}, \mathbf{f} \rangle \right|^2 = \mathbf{f}^* \mathbf{a}^{(j)} (\mathbf{a}^{(j)})^* \mathbf{f} = \text{Tr}(\mathbf{A}^{(j)} \mathbf{f} \mathbf{f}^*) \quad (3.10)$$

Furthermore, define  $\mathcal{A}$  be a linear operator which maps any positive semi-definite matrices  $\mathbf{F}$ , denoted by  $\mathbf{F} \geq 0$ , into

$$\mathcal{A}(\mathbf{F}) := (\text{Tr}(\mathbf{A}^{(1)} \mathbf{F}), \dots, \text{Tr}(\mathbf{A}^{(N)} \mathbf{F}))^T. \quad (3.11)$$

Therefore, the phase retrieval problem (3.9) is equivalent to

$$\begin{aligned} & \text{find } \mathbf{F} \\ & \text{subject to } \mathcal{A}(\mathbf{F}) = \tilde{\mathbf{f}}, \quad \mathbf{F} \geq 0, \quad \text{rank}(\mathbf{F}) = 1. \end{aligned} \quad (3.12)$$

Furthermore, phase retrieval (3.12) is equivalent to find a rank-one positive semi-definite matrix  $F$  satisfying the affine measurement constraints as follows,

$$\begin{aligned} \min \quad & \text{rank}(F) \\ \text{subject to} \quad & \mathcal{A}(F) = \tilde{f}, \quad F \geq 0. \end{aligned} \quad (3.13)$$

However, the rank-minimization problem (3.13) is still difficult. Thus the trace norm is utilized as a substitution for the rank functional in (3.13), which leads to the convex surrogate [26] and gives the following problem:

$$\begin{aligned} \min \quad & \text{Tr}(F) \\ \text{subject to} \quad & \mathcal{A}(F) = \tilde{f}, \quad F \geq 0. \end{aligned} \quad (3.14)$$

There are various solvers such as Nesterov accelerated first-order method [27] to handle the convex semi-definite programming problem (3.14). More profound results over the relationship between (3.13) and (3.14) is engaged in [28, 29]. For simplicity of implementation, we try to adopt a numerical solver TFOCS [30] which is a library of MATLAB files devised to facilitate the construction of optimal first-order methods for various convex optimization problems including the semi-definite programming problem (3.14). Hence, we choose to transform (3.14) into the following form

$$\begin{aligned} \min \quad & -\ell(\tilde{f}; \mathcal{A}(F)) + \lambda \text{Tr}(F), \\ \text{subject to} \quad & F \geq 0. \end{aligned} \quad (3.15)$$

where  $\ell(\tilde{f}; \mathcal{A}(F))$  is a smooth and concave (in  $F$ ) log-likelihood,  $\lambda$  is a parameter be chosen later. To solve the above composite program, we introduce a proximal gradient method to solve a general optimization problem:

$$\min \quad \phi(f) := g(f) + h(f), \quad (3.16)$$

where  $g$  is smooth and  $g, h$  are convex. Suppose  $\|\cdot\|$  is a chosen norm, the step size  $t_k$  is bounded away from zero and  $\nabla g$  satisfies a generalized Lipschitz criterion. Then we take the following iteration strategy with a generalized projection (cf [30]):

$$f_{k+1} = \arg \min_f \left[ g(f_k) + \langle \nabla g(f_k), f - f_k \rangle + h(f) + \frac{1}{2t_k} \|f - f_k\|^2 \right].$$

The advantage of above strategy is that global convergence is guaranteed. Moreover, the TFOCS solver contains a lot of optimal first-order variants based on a variation of the method described by Nesterov in [27]. Here, we choose a variant called the Auslender and Teboulle method [31].

## 4 | Numerical Examples

In this section, we present the numerical schemes and results for both the direct and inverse problems. Three examples are designed to demonstrate the effectiveness of proposed methods. Here,  $H = \frac{1}{2}$ , and  $\dot{B}^H(t) = \dot{W}_t$  represents the white noise.

For the forward problem, we use finite difference method to discretize the initial boundary value problem of the stochastic differential equation over the interval  $(0, L) \times (0, T)$  for some  $L, T > 0$ . Define the partition of spatial and time intervals with nodes  $x_i = ih_x, t_n = nh_t$  with  $h_x = \frac{L}{N_x}, h_t = \frac{T}{N_t}$  for  $n = 0, 1, \dots, N_t, i = 0, 1, \dots, N_x$ . Let  $u_i^n$  be approximation to  $u(x_i, t_n)$ . Then  $\partial_{tt}u, \partial_{xx}u$  and  $\partial_t u$  at  $(x_i, t_n)$  are approximated by central difference method. Moreover, the white noise  $\dot{W}_t$  at  $t = t_n$  is approximate by the increment of standard Brownian motion, that is,  $\dot{W}_t \approx \frac{W(t_{n+1}) - W(t_n)}{h_t} \stackrel{d}{=} \frac{\xi_n}{\sqrt{h_t}}$ , where  $a \stackrel{d}{=} b$  means the random variables  $a$  and  $b$  have the same distribution and  $\{\xi_i\}_{i=0}^{N_x-1}$  is a set of independent and identically distributed standard normal random variables, denoted by  $\xi_i \sim N(0, 1)$ . Assume the governing equation in (1.1) holds at the collocation points  $(x_i, t_n)$  gives

$$\begin{aligned} \left(1 + \frac{h_t}{2} a(t_n)\right) u_i^{n+1} &= h_t f(x_i) (W(t_{n+1}) - W(t_n)) \\ &+ \left(-1 + \frac{h_t}{2} a(t_n)\right) u_i^{n-1} + (2 - 2\sigma) u_i^n + \sigma(u_{i+1}^n + u_{i-1}^n), \end{aligned}$$

where  $\sigma = \frac{h_x^2}{h_x^2}$ . Furthermore, the boundary condition gives  $u(0, t) = 0 = u_0^n$  and  $u(L, t) = 0 = u_{N_x}^n$  for  $n = 0, 1, \dots, N_t$  and the initial conditions gives  $u(x_i, 0) = p(x_i) = u_i^0$  and  $\partial_t u(x_i, 0) = q(x_i) \approx \frac{u_i^1 - u_i^0}{2h_t}$  for  $i = 1, \dots, N_x - 1$ . Thus we obtain an explicit numerical scheme has a matrix system the dimension of which is  $(N_x + 1)(N_t + 1)$ . Next, we proceed to numerical schemes for the inverse problem. To recover initial status and random source simultaneously, we require the mean and the variance of the sample paths in two different times at  $T_1$  and  $T_2$  with two random mask  $M_1, M_2$  on the source  $f$ , respectively.

For initial value problem, if we can get the data  $u(x, t)$  at two different times  $T_1$  and  $T_2$ , we derive

$$\begin{aligned}\phi_{n,1}(T_1)\hat{p}(\omega_n) + \phi_{n,2}(T_1)\hat{q}(\omega_n) &= \mathbb{E}[\hat{u}(\omega_n, T_1)], \\ \phi_{n,1}(T_2)\hat{p}(\omega_n) + \phi_{n,2}(T_2)\hat{q}(\omega_n) &= \mathbb{E}[\hat{u}(\omega_n, T_2)].\end{aligned}\quad (4.1)$$

Then the initial value function  $p(x)$  and  $q(x)$  can be determined by solving the second order linear equations and the inverse Fourier transformation. To reconstruct  $p(x)$  and  $q(x)$ , the regularization method is needed in the numerical experiments, the numerical result is shown in the following examples where  $T_1 = \pi$  and  $T_2 = \frac{\pi}{2}$ .

For random source problem, the modulus of the Fourier coefficient  $|\hat{f}|$  can be reconstructed from (3.6) by utilizing the variance  $\mathbb{V}[\hat{u}(\omega_n, T)]$  of the Fourier transform of the field  $u(x, T)$  and the numerical integration of  $|g_n(T, \tau)|^2$ , that is,

$$|\hat{f}(\omega_n)| = \left( \frac{\mathbb{V}[\hat{u}(\omega_n, T)]}{\int_0^T |g_n(T, \tau)|^2 d\tau} \right)^{\frac{1}{2}}.$$

Although the variance of the DST data  $\mathbb{V}[u_k(T)]$  and the variance of the DFT data  $\mathbb{V}\hat{u}(\omega_n, T)$  both can be easily obtained from either direct problem or actual data in practice, the numerical integration still need calculation.

Notice  $g_k(T, \tau) = [-\phi_{k,1}(T)\phi_{k,2}(\tau) + \phi_{k,2}(T)\phi_{k,1}(\tau)]e^{\int_0^\tau a(s)ds}\chi_{\{\tau \leq T\}}$ , we first compute  $\phi_{k,1}, \phi_{k,2}$  from fundamental matrix  $\Phi_k = \begin{bmatrix} \phi_{k,1}, \phi_{k,2} \\ \phi'_{k,1}, \phi'_{k,2} \end{bmatrix}$ . The initial problem for the fundamental matrix  $\Phi_k$  is solved by the classic fourth-order Runge-Kutta formula:

$$\begin{aligned}K_1 &= h_t M_k(t_j) \Phi_{k,j}, \\ K_2 &= h_t M_k\left(t_j + \frac{h_t}{2}\right) \left(\Phi_{k,j} + \frac{1}{2} K_1\right), \\ K_3 &= h_t M_k\left(t_j + \frac{h_t}{2}\right) \left(\Phi_{k,j} + \frac{1}{2} K_2\right), \\ K_4 &= h_t M_k(t_j + h_t) (\Phi_{k,j} + K_3), \\ \Phi_{k,j+1} &= \Phi_{k,j} + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4).\end{aligned}$$

For  $j = 0, 1, \dots, N_t$ , where  $M_k(t) = \begin{bmatrix} 0 & 1 \\ -\lambda_k & -a(t) \end{bmatrix}$ , and  $\Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

Knowing  $\phi_{k,1}$  and  $\phi_{k,2}$ , we can compute the integration of  $|g_k(T, \tau)|^2$  by numerical integration method such as trapezoidal rule or Simpson's rule. By the way, we can avoid computing  $e^{\int_0^\tau a(s)ds}$  by using the fact  $\det \Phi(\tau) = e^{-\int_0^\tau a(s)ds}$  and the previous calculation results of  $\Phi$ . It should be noted that the time partition of  $\Phi$  can be different from the direct problem, but we use the same notation here for simplicity.

Next we try solve the phase retrieval problem numerically, provided more measurements by phaselift with two masks. Let  $M_1$  and  $M_2$  be two masks with  $M_1$  being the  $(N_x + 1) \times (N_x + 1)$  identity matrix and  $M_2$  being a random binary mask. Using these masks  $M_i$  to perform the structured illumination with source  $M_i f$ , we can obtain  $2(N_x + 1)$  intensity measurements  $\mathcal{A}(f f^*)$ , where the operator  $\mathcal{A}$  is defined in (3.11) with  $N = 2(N_x + 1)$ , the vectors  $\{\mathbf{a}^{(m)}\}_{m=1}^{2(N_x+1)}$  involved

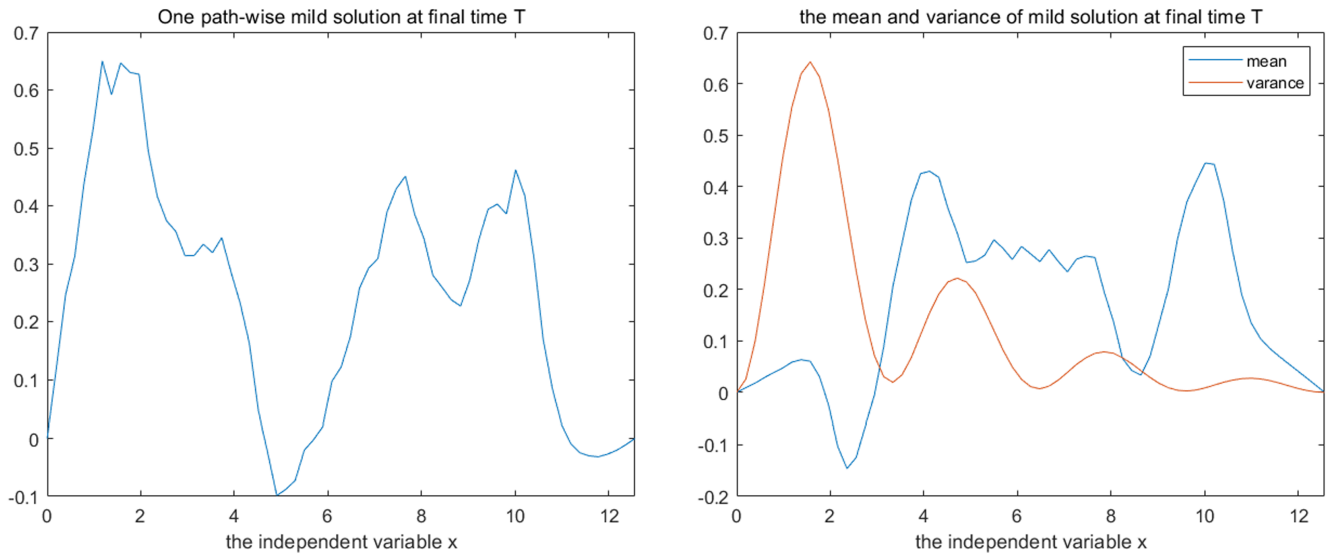


**FIGURE 1** | Example1: One path-wise mild solution is given in left and the mean and variance of solution is shown in right.

in the operator  $\mathcal{A}$  are defined by

$$\mathbf{a}^{(m)} = \begin{cases} \mathbf{M}_1 \mathbf{h}^{(m)}, & m = 1, \dots, N_x + 1, \\ \mathbf{M}_2 \mathbf{h}^{(m-N_x-1)}, & m = N_x + 2, \dots, 2(N_x + 1). \end{cases}$$

To recover the source  $\mathbf{f}$ , it suffices to solve the convex optimization problem mentioned in Section 3.2 based on the objective functional

$$\phi(\mathbf{f}) = g(\mathbf{f}) + h(\mathbf{f}) = \frac{1}{2} \|\mathbf{b} - \mathcal{A}(\mathbf{f} \mathbf{f}^*)\|_2^2 + h(\mathbf{f}),$$

where  $\|\cdot\|_2$  denotes the Euclidean norm and  $h$  is the indicator function defined by

$$h(\mathbf{f}) = \begin{cases} \lambda \text{Tr}(\mathbf{f} \mathbf{f}^*), & \text{if } \mathbf{f} \mathbf{f}^* \text{ is Hermitian positive semi-definite,} \\ \infty, & \text{else.} \end{cases}$$

Here  $\lambda > 0$  is a regularization parameter, and  $\mathbf{b}$  is the quadratic Fourier modulus obtained by the measurement  $\{u_i^{N_t}\}_{i=1}^{2(N_x+1)}$  at  $t = T$  where the first half is perturbed by  $\mathbf{M}_1 \mathbf{f}$  and the second half is perturbed by  $\mathbf{M}_2 \mathbf{f}$  with masks. More precisely,  $\mathbf{b} = (b_1, \dots, b_{2(N_x+1)})^\top$  has entries

$$b_i = \frac{\mathbb{V}[\hat{u}_i^{N_t}]}{\int_0^T |g_{k_i}(T, \tau)|^2 d\tau}, \quad i = 1, 2, \dots, 2(N_x + 1),$$

where  $(\hat{u}_1^{N_t}, \dots, \hat{u}_{2(N_x+1)}^{N_t})^\top$  is the combination of the  $(N_x + 1)$ -point DFT of the two parts of measurement  $(u_1^{N_t}, \dots, u_{2(N_x+1)}^{N_t})^\top$  and  $\lambda_{k_i} = \lambda_{k_{i+(N_x+1)}} = (\frac{2(i-1)}{L})^2$ ,  $i = 1, 2, \dots, N_x + 1$ . The Auslender and Teboulle method is applied to solve the above problem, which is terminated when the relative residual error of our reconstructed result  $|f|$  is less than a fixed tolerance or iteration steps reaches the upper bound. Particularly, if  $f(x)$  is a real-valued non-negative function, then the recovery is unique.

We implement the proposed method in the following three example to recover the three unknowns  $p, q, f$ . The effectiveness of the numerical method can be verified from following three examples.

**Example 1.** In this example, set  $a(t) = \frac{1+\cos(t)}{2}$  and select a narrowband symmetric function and a broadband symmetric function as the initial displacement  $p(x)$  and the initial velocity  $q(x)$  as follows, respectively. As for the random source,

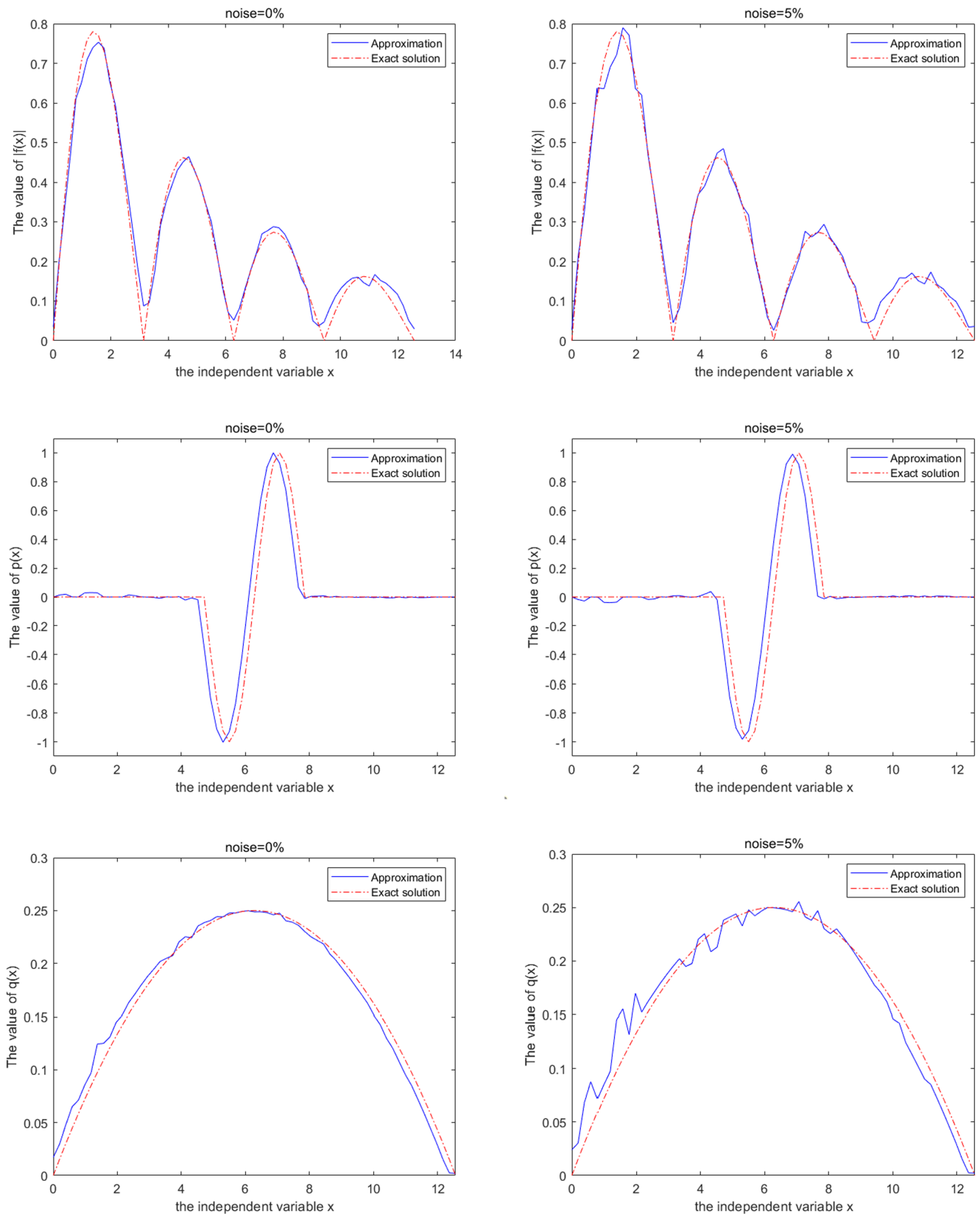


**FIGURE 2** | Example1: The exact solutions are plotted against the reconstructed solutions with different noise levels, where the damping term  $a(t) = \frac{1+\cos(t)}{2}$ .

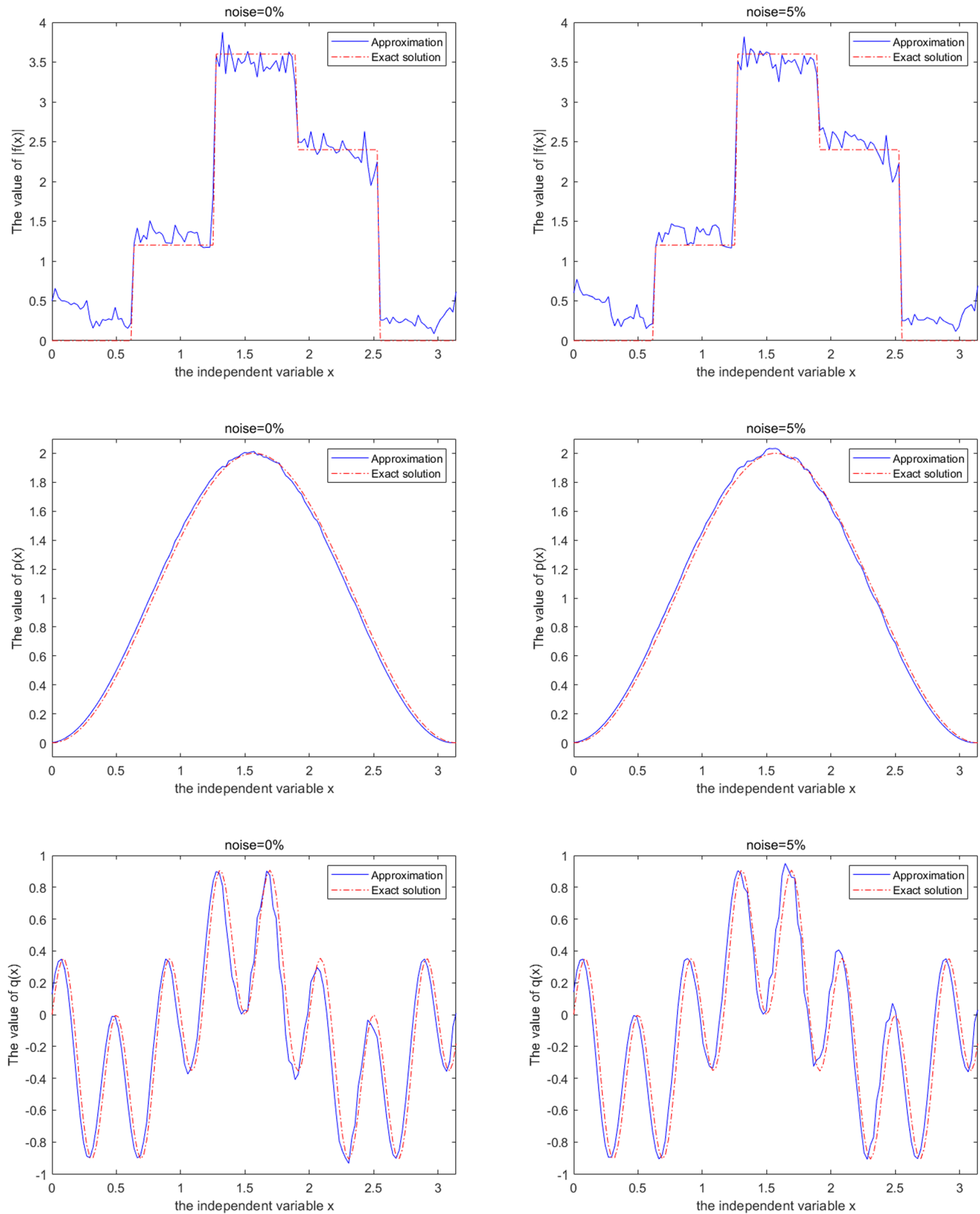


**FIGURE 3** | Example2: The exact solutions are plotted against the reconstructed solutions with different noise levels, where the damping term  $a(t) = |\sin(t)|e^{-2t}$ .

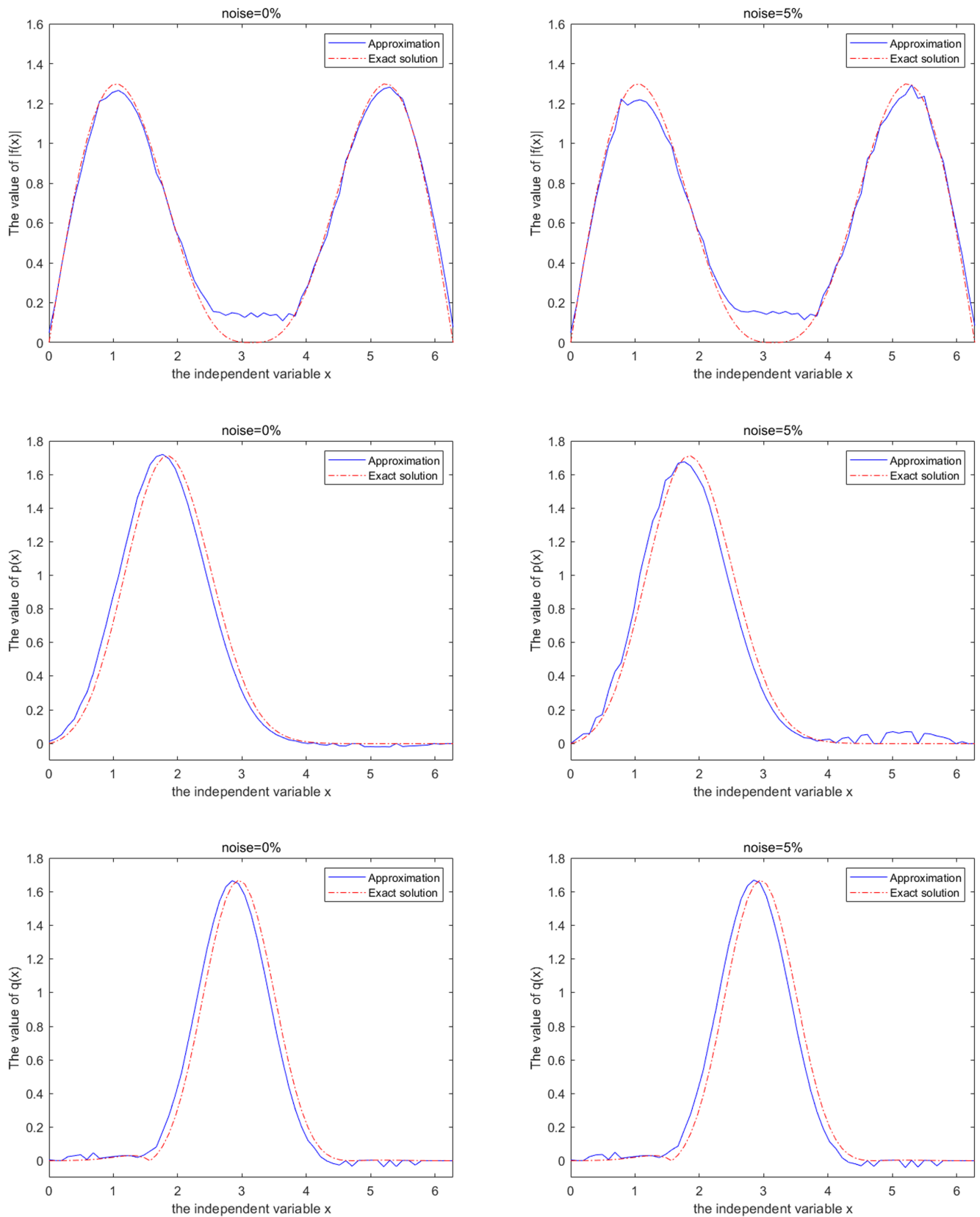


**FIGURE 4** | Example3: The exact solutions are plotted against the reconstructed solutions with different noise levels, where the damping term  $a(t) = |\sin(2t) \cos(3t)|$ .

we set a decaying smooth example where

$$p(x) = \begin{cases} \sin(2x), & \frac{3}{8}L \leq x \leq \frac{5}{8}L, \\ 0, & \text{else,} \end{cases} \quad q(x) = \frac{x}{L}(1 - \frac{x}{L}), \quad f(x) = \sin(x)e^{-\frac{x}{6}}.$$

In computation, we take  $L = 4\pi$ ,  $N_x = 64$ ,  $N_t = 64$ . The forward problem is to compute  $u(x, t)$  at different collocation points. One path-wise mild solution is given in left of Figure 1. Provided that there are 2000 paths, the mean and variance of solution is shown in right of Figure 1. Moreover, the corresponding inversion results are shown in Figure 2. The left column depicts the recovered random source  $|f(x)|$ , the initial conditions  $p(x)$  and  $q(x)$  with noisy free synthetic data. The right column shows the counter part of left column but with 5% noisy synthetic data. Figure 2 clearly shows that the inversion results converge to exact solution if noise level is decreasing. In the numerical experiment, we notice that the initial functions with narrow frequency has higher inversion accuracy.

**Example 2.** In this example, the initial value functions are taken to be  $p(x) = 1 - \cos(2x)$  and an oscillating function  $q(x) = \sin(2\pi x)\cos(3\pi x)$ . A discontinuous source function  $f$  is considered:

$$f(x) = \begin{cases} 0, & x \in [0, \frac{1}{5}L), \\ 1.2, & x \in [\frac{1}{5}L, \frac{2}{5}L), \\ 3.6, & x \in [\frac{2}{5}L, \frac{3}{5}L), \\ 2.4, & x \in [\frac{3}{5}L, \frac{4}{5}L), \\ 0, & x \in [\frac{4}{5}L, L], \end{cases}$$

where  $L = \pi$ . We can clearly observe that the reconstruction of the discontinuous source function has lower accuracy and stability. And the method for recovering initial values proves to be effective regardless of oscillatory or not.

**Example 3.** In the last example, the three unknowns is  $p(x) = xe^{(x-\frac{\pi}{2})^2}$ ,  $q(x) = |(x - \frac{3\pi}{2})\cos(x)|e^{(x-\pi)^2}$  and  $f(x) = (1 + \cos(x))\sin(x)$  with  $L = 2\pi$ , where the damping term is an oscillating function.

Different from the previous two examples, the source function of this example is symmetric while the initial value functions are asymmetric. We notice that the reconstruction of the source function has lower accuracy at the extreme point especially in the presence of noise. As for the reversion of the initial value functions, the error is more visible near the endpoint of the interval.

From the examples above, we can verify the effectiveness and robustness of the proposed numerical schemes. Clearly, the results of the proposed method to reconstruct initial values perform well. However, it can be observed from the numerical experiments that the proposed method of the reconstruction of the modulus of the source still has drawbacks especially for the discontinuous sources, which needs to be improved in future (Figures 3 and 4).

## 5 | Conclusion

In this work, we have conducted a study on an inverse problem with three unknowns involving the stochastic damped wave equation driven by a Gaussian random field over time. By examining the corresponding stochastic initial value problem in the frequency domain, we establish the well-posedness of the direct problem and the uniqueness of the solution to the inverse problem. To tackle the inverse problem, we divide it into an inverse initial value problem and an inverse random source problem. To address the inverse initial value problem, we require mean data from two distinct moments. Regarding the random source problem with white noise, we introduce a phase retrieval problem and employ the phaselift method along with additional measurements using random masks to recover the modulus of the source. Numerical examples are presented to demonstrate the sufficiency and effectiveness of this method in reconstructing the three unknowns.

This framework exhibits versatility, prompting further interest in exploring higher dimensional inverse random source problems. Additionally, the problem with random source driven by coloured noise is also worthy further research.

## Author Contributions

**Kuijian Chang:** data curation; formal analysis; visualization; writing – original draft. **Yuxuan Gong:** methodology; formal analysis. **Xiang Xu:** supervision; conceptualization; writing – review and editing; funding acquisition.

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## Conflicts of Interest

The authors declare no conflicts of interest.

## Data Availability Statement

I confirm that my Data Availability Statement complies with the Expects Data Policy.

## References

1. J. Sjöstrand, “Asymptotic Distribution of Eigenfrequencies for Damped Wave Equations,” *Publications of the Research Institute for Mathematical Sciences* 36, no. 5 (2000): 573–611.
2. Pierangelo Marcati and Kenji Nishihara, “The  $L^p$ - $L^q$  Estimates of Solutions to One-Dimensional Damped Wave Equations and Their Application to the Compressible Flow Through Porous Media,” *Journal of Differential Equations* 191, no. 2 (2003): 445–469.
3. K. Nishihara, “ $L^p$  –  $L^q$  Estimates of Solutions to the Damped Wave Equation in 3-Dimensional Space and Their Application,” *Mathematische Zeitschrift* 244, no. 3 (2003): 631–649.
4. V. Pata and S. Zelik, “A Remark on the Damped Wave Equation,” *Communications on Pure and Applied Analysis* 5, no. 3 (2006): 611–616.
5. V. Pata and M. Squassina, “On the Strongly Damped Wave Equation,” *Communications in Mathematical Physics* 253, no. 3 (2005): 511–533.
6. D. Borisov and P. Freitas, “Eigenvalue Asymptotics, Inverse Problems and a Trace Formula for the Linear Damped Wave Equation,” *Journal of Differential Equations* 247, no. 11 (2009): 3028–3039.
7. F. Colombo, “An Inverse Problem for the Strongly Damped Wave Equation With Memory,” *Nonlinearity* 20, no. 3 (2007): 659–683.
8. N. H. Tuan, A. Vo Van, and N. H. Can, “Regularization of Initial Inverse Problem for Strongly Damped Wave Equation,” *Applicable Analysis* 97, no. 1 (2018): 69–88.
9. N. H. Tuan, D. V. Nguyen, A. Vo Van, and D. Lesnic, “Recovering the Initial Distribution for Strongly Damped Wave Equation,” *Applied Mathematics Letters* 73 (2017): 69–77.
10. A. J. Devaney, “The Inverse Problem for Random Sources,” *Journal of Mathematical Physics* 20, no. 8 (1979): 1687–1691.
11. G. Bao, C. Chen, and P. Li, “Inverse Random Source Scattering Problems in Several Dimensions,” *SIAM/ASA Journal on Uncertainty Quantification* 4, no. 1 (2016): 1263–1287.
12. G. Bao, S.-N. Chow, P. Li, and H. Zhou, “An Inverse Random Source Problem for the Helmholtz Equation,” *Mathematics of Computation* 83, no. 285 (2014): 215–233.
13. X. Feng, M. Zhao, P. Li, and X. Wang, “An Inverse Source Problem for the Stochastic Wave Equation,” *Inverse Problems and Imaging* 16, no. 2 (2022): 397–415.
14. M. Li, C. Chen, and P. Li, “Inverse Random Source Scattering for the Helmholtz Equation in Inhomogeneous Media,” *Inverse Problems* 34, no. 1 (2017): 015003.
15. Y. Gong, X. Peijun Li, X. Wang, and X. Xu, “Numerical Solution of an Inverse Random Source Problem for the Time Fractional Diffusion Equation via Phaselift,” *Inverse Problems* 37, no. 4 (2021): 045001.
16. L. Kening and B. Schmalfuß, “Invariant Manifolds for Stochastic Wave Equations,” *Journal of Differential Equations* 236, no. 2 (2007): 460–492.

17. M. Yang, J. Duan, and P. Kloeden, "Asymptotic Behavior of Solutions for Random Wave Equations With Nonlinear Damping and White Noise," *Nonlinear Analysis: Real World Applications* 12, no. 1 (2011): 464–478.
18. T. B. Ngoc, T. N. Thach, D. O'Regan, and N. H. Tuan, "On Inverse Initial Value Problems for the Stochastic Strongly Damped Wave Equation," *Applicable Analysis* 101, no. 2 (2022): 527–544.
19. Q. Lü and X. Zhang, "Global Uniqueness for an Inverse Stochastic Hyperbolic Problem With Three Unknowns," *Communications on Pure and Applied Mathematics* 68, no. 6 (2015): 948–963.
20. E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voroninski, "Phase Retrieval via Matrix Completion," *SIAM Review* 57, no. 2 (2015): 225–251.
21. C.-C. Chen, J. Miao, C. W. Wang, and T. K. Lee, "Application of Optimization Technique to Noncrystalline x-Ray Diffraction Microscopy: Guided Hybrid Input-Output Method," *Physical Review B* 76, no. 6 (2007): 064113.
22. J. R. Fienup, "Reconstruction of an Object From the Modulus of Its Fourier Transform," *Optics Letters* 3, no. 1 (1978): 27–29.
23. I. Johnson, K. Jefimovs, O. Bunk, et al., "Coherent Diffractive Imaging Using Phase Front Modifications," *Physical Review Letters* 100, no. 15 (2008): 155503.
24. E. G. Loewen and E. Popov, *Diffraction Gratings and Applications* (CRC Press, 2018).
25. A. Faridian, D. Hopp, G. Pedrini, U. Eigenthaler, M. Hirscher, and W. Osten, "Nanoscale Imaging Using Deep Ultraviolet Digital Holographic Microscopy," *Optics Express* 18, no. 13 (2010): 14159–14164.
26. M. Mesbahi and G. P. Papavassilopoulos, "On the Rank Minimization Problem Over a Positive Semidefinite Linear Matrix Inequality," *IEEE Transactions on Automatic Control* 42, no. 2 (1997): 239–243.
27. Y. Nesterov, "Nonsmooth Convex Optimization," in *Introductory Lectures on Convex Optimization: A Basic Course, Volume 87* (Springer Science & Business Media, 2003).
28. E. J. Candès and B. Recht, "Exact Matrix Completion via Convex Optimization," *Foundations of Computational Mathematics* 9, no. 6 (2009): 717–772.
29. E. J. Candès and T. Tao, "The Power of Convex Relaxation: Near-Optimal Matrix Completion," *IEEE Transactions on Information Theory* 56, no. 5 (2010): 2053–2080.
30. S. R. Becker, E. J. Candès, and M. C. Grant, "Templates for Convex Cone Problems With Applications to Sparse Signal Recovery," *Mathematical Programming Computation* 3, no. 3 (2011): 165–218.
31. A. Auslender and M. Teboulle, "Interior Gradient and Proximal Methods for Convex and Conic Optimization," *SIAM Journal on Optimization* 16, no. 3 (2006): 697–725.