

Uniqueness of recovering random perturbed potentials of biharmonic operator from the far-field pattern

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Abstract

This paper is concerned with the inverse scattering for the biharmonic operator Δ^2 with a stochastic first-order perturbation $A(x) \cdot \nabla + V(x)$. The vector-valued potential A and the scalar-valued potential V are both assumed to be generalized microlocally isotropic Gaussian random fields with classical pseudodifferential covariance operators. The goal is to determine their principal symbols simultaneously from the observed far-field scattering pattern. This problem is challenging due to the roughness of first-order perturbation and the higher order elliptic operator. The key ingredient to overcome the challenge is to employing scattering theory, which results in the analytic domain and an upper bound for the resolvent of the fourth-order elliptic operator with rough potentials. As a consequence, by analyticity of the resolvent, uniqueness can be achieved by far-field pattern measured only in a finite interval of frequencies. Moreover, by ergodicity it is shown that the principal symbols of the covariance operators of the two random potentials can be uniquely reconstructed simultaneously by a single realization of the far-field pattern averaged over the high frequency band almost surely. In addition, for the first time, reconstruction formulas are established on inverse scattering for elliptic operators with a random first-order perturbation.

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1. Introduction

Inverse scattering problems aim at determining scatterers from appropriate measurements. They arise in many scientific and industrial areas including quantum physics, biomedical imaging, radar and sonar, and geophysical exploration. In many applications which involve uncertainties, one usually assumes that the scatterer can be described by random process, and hence the scattered field is also random. Recently, random scattering problems have received considerable attention due to their significant applications in situations involving unpredictable environments, incomplete knowledge of the system, and random measurement noise. Compared with their deterministic counterparts, stochastic inverse problems are more difficult to handle. A feature of such problems is that sometimes the scatterer is too rough to exist pointwisely and thus should be understood in the sense of distribution. Moreover, in these situations, it is less meaningful to characterize the full scatterer but to determine their statistical properties such as mean and variance which are used to quantify the uncertainties.

This paper is concerned with the scattering problem for the perturbed biharmonic operator with the Sommerfeld radiation condition

$$\begin{cases} (\Delta^2 - k^4 + \mathbf{A} \cdot \nabla + V)u = 0, & \text{in } \mathbb{R}^3, \\ u = u^{inc} + u^{sc}, \\ \lim_{r \rightarrow \infty} r(\partial_r u^{sc} - iku^{sc}) = 0, & r = |x|, \end{cases} \quad (1.1)$$

where $k > 0$ is the wavenumber, u is the total wave field, u^{sc} is the scattered wave field and $u^{inc} := e^{ikx \cdot d}$ is the incident plane wave with direction $d \in \mathbb{S}^2$. The vector-valued potential $\mathbf{A}$ and the scalar-valued potential V are both assumed to be random fields with compact supports.

In this work, we assume that both $\mathbf{A}$ and V are generalized microlocally isotropic Gaussian (GMIG) random fields supported in a bounded Lipschitz domain $D \subset \mathbb{R}^3$. The GMIG random fields encompass various important random fields, such as white noise, Markov fields and fractional Brownian motion [9]. Mathematically, the covariance operator $C_{\mathbf{A}}$ of the potential $\mathbf{A}$ is assumed to be a classical pseudodifferential operator of order $-m_{\mathbf{A}}$, and the principal symbol of $C_{\mathbf{A}}$ has the form $\mathbb{H}(x)|\xi|^{-m_{\mathbf{A}}}$, where $\mathbb{H}(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ is a smooth matrix-valued function with $\text{supp } \mathbb{H} \subset \subset D$. The matrix $\mathbb{H}$ is called the micro-correlation strength of $\mathbf{A}$. Similarly, V is also assumed to be a GMIG of order $-m_V$ and the principal symbol of its covariance is $h(x)|\xi|^{-m_V}$ with $h \in C_0^\infty(\mathbb{R}^3; \mathbb{R})$ and $\text{supp } h \subset \subset D$. Detailed information on GMIG and its properties are given in Section 2. The main goal of this paper is to reconstruct both the micro-correlation strengths $\mathbb{H}$ and h from the far-field pattern associated with (1.1).

The inverse random potential scattering problems driven by the GMIG random field were considered in [2,9–11] for the second-order Schrödinger and elastic wave equations. In these works, it was shown that the strength of the random potential can be uniquely reconstructed by a single realization of the near- or far-field data almost surely. The analysis employs the method of using Born series expansion, the limiting absorption principle for the Schrödinger equation and microlocal analysis developed in [9]. Later, similar results were obtained in [12] for the biharmonic wave equation. In all the aforementioned works, the potential functions are assumed

to be a zeroth-order perturbation of the elliptic equation. To our knowledge, the inverse scattering problems for elliptic operators with a stochastic first-order perturbation remain wide open.

In the deterministic settings, there exist some works on the inverse problems of the first-order perturbed biharmonic operator due to their applications in the theory of quantum physics, vibrations of elasticity and beams [5]. For the inverse boundary problem, [7,8] showed that the first-order perturbation can be uniquely determined from two sets of DtN maps. For the inverse scattering problem, the authors in [16] proved that the combination $-\frac{1}{2}\nabla \cdot \mathbf{A} + V$ of the potential functions can be uniquely determined from the far-field data over an infinite high frequency band. Further, in a recent work [17], the authors derived increasing stability estimates of recovering the divergence-free part of $\mathbf{A}$ and the combination $-\frac{1}{2}\nabla \cdot \mathbf{A} + V$ from the multi-frequency far-field data, respectively. Moreover, they showed the uniqueness of recovering both $\mathbf{A}$ and V by partial far-field data based upon scattering theory and an upper bound for the resolvent of the fourth order elliptic operator. However, the analysis in these works relies on Helmholtz decomposition of the vector potential $\mathbf{A}$, which is not applicable to stochastic case.

In this work, we are concerned with determining the micro-correlation strengths of the two random potentials simultaneously. The main results (cf. Theorem 4.1 and 4.2) show that both the micro-correlation strengths $\mathbb{H}$ and h can be uniquely reconstructed by the high frequency limit of the correlation of the far-field pattern from a single realization of the random potentials almost surely by ergodicity. Consequently, by analyticity of the resolvent with respect to complex wavenumber, the uniqueness can indeed be obtained by far-field pattern measured only in a finite interval of frequencies (cf. Theorem 4.3). Apparently, due to the existence of the vector-valued random potential function, the inverse problem considered in this work is much more complicated compared with the previous ones of only determining the zeroth-order perturbation. For instance, instead of being a scalar-valued function, the strength $\mathbb{H}$ of the principal symbol for the random vector potential is now a matrix function. Moreover, the problem of determining both $\mathbb{H}$ and h is more challenging.

The investigation of the inverse problem requires resolvent estimates for the perturbed biharmonic operator with sufficient decay with respect to the wavenumber, which allows the expansion of the far-field pattern into Born series. To this end, we employ the scattering theory introduced in [4] to investigate the resolvent with rough first-order perturbation. Specifically, by investigating the meromorphic continuation of the resolvent, a logarithmic analytic domain and useful resolvent estimates are obtained. In comparison with the limiting absorption principle utilized in [9], the scattering theory seems to be more convenient to use in inverse scattering problems. In general, limiting absorption principle only provides resolvent estimates on the real axis in weighted Sobolev space, and the resolvent may not be further extended to the lower-half complex plane [1,6,14]. In contrast, scattering theory shows that meromorphic continuation of the resolvent is possible by considering the resolvent as a family of operators of $L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}$. As a consequence, resolvent estimates and an analytic domain can be obtained. Thus, by analyticity of the resolvent, one may only use multi-frequency data in a finite interval other than requiring data at all high frequencies to achieve the uniqueness. Since the scatterer in inverse scattering usually has compact support, and the data is required to be measured on compact sets in practical applications, the application of scattering theory is feasible in this setting.

Another difficulty for the inverse problem is that, as the micro-correlation strength of the vector random potential is a matrix function, the backscattering data used in [2] for the recovery of the zeroth-order potential is no longer applicable. To fix this issue, we employ far-field pattern with appropriately chosen incident and observation directions, which reveals an explicit relation between the vector potential and the far-field pattern. Based on this relation, a reconstruction

formula for the strength $\mathbb{H}$ of the vector potential is derived by the correlation far-field pattern. Next, we continue to reconstruct h for the scalar potential from the backscattering far-field pattern. In this step, we establish an explicit reconstruction formula for h by the far-field pattern and $\mathbb{H}$. Both the reconstruct formulas use a single realization of the far-field pattern by ergodicity with probability one. Compared with the deterministic setting, by utilizing the correlation data we do not resort to the Helmholtz decomposition of the vector potential.

This paper is outlined as follows. In Section 2 we introduce the definition and properties of the generalized microlocally isotropic Gaussian random fields. Section 3 is devoted to the meromorphic continuation of the perturbed biharmonic operator. A logarithmic analytic domain and resolvent estimates are derived for the resolvent. In Section 4, we introduce the far-field pattern and first prove the uniqueness of recovering the micro-correlation strength of the vector potential function without knowing the scalar potential. Then, we show that the micro-correlation strength of the scalar potential can also be uniquely determined given the micro-correlation strength of the vector potential. Uniqueness of determining both the micro-correlation strengths of the two random potentials by far-field pattern measured only in a finite interval of frequencies is derived by the analyticity of the resolvent.

2. Preliminaries

In this section, we introduce vector-valued and scalar-valued GMIG random fields to characterize the random potentials considered in this paper.

Denote the function spaces $W^{s,p} := W^{s,p}(\mathbb{R}^3)$ as the standard Sobolev spaces. Let $H^s := W^{s,2}$. The function spaces $H_{loc}^s := H_{loc}^s(\mathbb{R}^3)$ and $W_{comp}^{s,p} := H_{comp}^s(\mathbb{R}^3)$ are respectively defined as follows

$$\begin{aligned} H_{loc}^s(\mathbb{R}^3) &:= \{u \in \mathcal{D}'(\mathbb{R}^3) : \chi u \in H^s(\mathbb{R}^3), \forall \chi \in C_0^\infty(\mathbb{R}^3)\}, \\ H_{comp}^s(\mathbb{R}^3) &:= \{u \in H^s(\mathbb{R}^3) : \exists \chi \in C_0^\infty(\mathbb{R}^3), \chi u = u\}, \end{aligned}$$

where $\mathcal{D}'$ is the dual space of $\mathcal{D} := C_0^\infty(\mathbb{R}^3)$. Let $\mathcal{L}(X, Y)$ be the space which consists of all bounded linear operators from the function space X to the space Y . Moreover, the space $\mathcal{L}(X, Y)$ is equipped with the operator norm $\|\cdot\|_{\mathcal{L}(X, Y)}$. Let $a \lesssim b$ denote $a \leq Cb$, where $C > 0$ is a generic constant whose specific value is not required but should be clear from the context.

Now we introduce the real-valued generalized Gaussian random field. Let $\mathcal{D}(\mathbb{R}^3; \mathbb{R})$ stand for the real-valued test function space and its dual space is denoted as $\mathcal{D}'(\mathbb{R}^3; \mathbb{R})$. Given a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a real-valued random field $g : \Omega \rightarrow \mathcal{D}'(\mathbb{R}^3; \mathbb{R})$ is called a generalized Gaussian random field if $\omega \mapsto \langle g(\omega), \phi \rangle$ is a Gaussian random variable for all $\phi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})$. The domain of $g(\omega)$ can be extended to the space of complex-valued smooth functions with compact support as

$$\langle g(\omega), \phi \rangle = \langle g(\omega), \Re \phi \rangle + i \langle g(\omega), \Im \phi \rangle, \quad \phi \in \mathcal{D}.$$

The expectation $\mathbb{E}g \in \mathcal{D}'(\mathbb{R}^3; \mathbb{R})$ and covariance $\text{Cov } g : \mathcal{D}(\mathbb{R}^3; \mathbb{R}) \times \mathcal{D}(\mathbb{R}^3; \mathbb{R}) \rightarrow \mathbb{R}$ are respectively defined by

$$\begin{aligned} \mathbb{E}g : \phi &\mapsto \mathbb{E}\langle g, \phi \rangle, \quad \phi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}), \\ \text{Cov } g : (\phi_1, \phi_2) &\mapsto \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}), \end{aligned}$$

where

$$\text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle) = \mathbb{E}[(\langle g, \phi_1 \rangle - \mathbb{E}\langle g, \phi_1 \rangle)(\langle g, \phi_2 \rangle - \mathbb{E}\langle g, \phi_2 \rangle)].$$

Define the covariance operator C_g by

$$\langle C_g \phi_1, \phi_2 \rangle = \text{Cov}(\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle), \quad \phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}).$$

Let $K_g(x, y)$ be the Schwarz kernel of C_g , i.e.,

$$\langle C_g \phi_1, \phi_2 \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \phi_2(y) K_g(x, y) \phi_1(x) \, dx dy.$$

In this paper, we consider the generalized Gaussian random fields with zero expectation defined as follows.

Definition 2.1. A generalized Gaussian random field g with zero expectation is said to be microlocally isotropic of order $-m \in \mathbb{R}$ in $D \subset \mathbb{R}^3$ if its covariance operator C_g is a classical pseudodifferential operator of order $-m$ with the principal symbol $\mu(x)|\xi|^{-m}$, where $\mu(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{R})$ is a real-valued function with $\text{supp } \mu \subset\subset D$. Here μ is called the micro-correlation strength of g .

Let c_g be the symbol of the operator C_g , i.e.,

$$C_g(\phi)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} c_g(x, \xi) \hat{\phi}(\xi) \, d\xi, \quad \phi \in \mathcal{D},$$

which gives

$$K_g(x, y) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(x-y) \cdot \xi} c_g(x, \xi) \, d\xi.$$

The following lemma concerns the regularity of the microlocally isotropic Gaussian random field.

Lemma 2.1. Let g be a microlocally isotropic Gaussian random field of order $-m$ in D . Then $g \in W^{\frac{m-3}{2}-\delta, p}(D)$ almost surely for all $\delta > 0$ and $1 < p < \infty$.

Throughout this paper, we consider GMIG random fields of order $-m$ with $m > 2$. In this case, there holds the following results for the kernel.

Lemma 2.2. Let g be a GMIG random field of order $-m$ in D . The covariance function $K_g(x, y)$ has the following form:

(i) If $a < m - 2 \leq a + 1$ with $a = 1, 2, 3, \dots$,

$$K_g(x, y) = c\mu(x)|x - y|^{m-3} + F_m(x, y),$$

where c is a constant dependent on m and $F_m(x, y) \in C^{a,\gamma}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ with $\gamma \in (0, m-2-a)$.
(ii) If $2 < m < 3$,

$$K_g(x, y) = c\mu(x)|x - y|^{m-3} + F_m(x, y),$$

where c is a constant dependent on m and $F_m(x, y) \in C^{0,\gamma}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ with $\gamma \in (0, m-2)$.
(iii) If $m = 3$,

$$K_g(x, y) = c\mu(x) \log|x - y| + F_m(x, y),$$

where c is a constant dependent on $F_m(x, y) \in C^{0,\gamma}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})$ with $\gamma \in (0, 1)$.

Now we consider vector fields. A vector field $\mathbf{g} = (g_1, g_2, g_3)^T$ is said to be a real-valued GMIG random field of order $-m$ in D if each component g_j , $j = 1, 2, 3$, is a real-valued GMIG random field of the same order $-m$ in D . Thus, the expectation $\mathbb{E}\mathbf{g} \in \mathcal{D}'(\mathbb{R}^3; \mathbb{R})^3$ and covariance $\text{Cov } \mathbf{g} : \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3 \times \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3 \rightarrow \mathbb{R}$ are defined as follows

$$\begin{aligned} \mathbb{E}\mathbf{g} : \boldsymbol{\phi} &\mapsto \mathbb{E}\langle \mathbf{g}, \boldsymbol{\phi} \rangle, \quad \boldsymbol{\phi} \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3, \\ \text{Cov } \mathbf{g} : (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2) &\mapsto \text{Cov}(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle), \quad \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3. \end{aligned}$$

The covariance operator C_g is defined by

$$\langle C_g \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = \text{Cov}(\langle \mathbf{g}, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{g}, \boldsymbol{\phi}_2 \rangle), \quad \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})^3.$$

Let the 3×3 matrix-valued function $K_g(x, y)$ be the Schwarz kernel of C_g , i.e.,

$$\langle C_g \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \boldsymbol{\phi}_2(y)^T K_g(x, y) \boldsymbol{\phi}_1(x) dx dy.$$

As $\mathbf{g}$ is microlocally isotropic of order $-m$, the covariance operator C_g is a classical pseudo-differential operator with the principal symbol $\mathbb{M}(x)|\xi|^{-m}$, where $\mathbb{M}(x) \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ is a smooth matrix-valued function with $\text{supp } \mathbb{M} \subset\subset D$. Here $\mathbb{M}$ is similarly called the micro-correlation strength of $\mathbf{g}$.

3. Meromorphic continuation and resolvent estimates

In this section, we investigate the meromorphic continuation and estimates of the resolvent of the biharmonic operator. As a consequence, we establish the well-posedness of the direct scattering problem.

Denote the resolvent of the free Schrödinger operator and the resolvent of the free biharmonic operator by $\tilde{R}_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ and $R_0(\lambda) = (\Delta^2 - \lambda^4)^{-1}$, respectively. From the decomposition

$$(\Delta^2 - \lambda^4)^{-1} = \frac{1}{2\lambda^2} ((-\Delta - \lambda^2)^{-1} - (-\Delta + \lambda^2)^{-1})$$

we have

$$R_0(\lambda) = \frac{1}{2\lambda^2}(\tilde{R}_0(\lambda) - \tilde{R}_0(i\lambda)). \quad (3.1)$$

The following lemma concerns the analyticity and estimates of the free resolvent $\tilde{R}_0(\lambda)$ for the Schrödinger operator.

Lemma 3.1. *The free resolvent $\tilde{R}_0(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $\Im \lambda > 0$ as a family of operators*

$$\tilde{R}_0(\lambda) : H^s(\mathbb{R}^3) \rightarrow H^t(\mathbb{R}^3),$$

for all $s \in \mathbb{R}$ and $t \in [s, s+2]$. Furthermore, given a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$, $\tilde{R}_0(\lambda)$ can be extended to a family of analytic operators for $\lambda \in \mathbb{C}$ as follows

$$\chi \tilde{R}_0(\lambda) \chi : H^s(\mathbb{R}^3) \rightarrow H^t(\mathbb{R}^3)$$

with the resolvent estimates

$$\|\chi \tilde{R}_0(\lambda) \chi\|_{\mathcal{L}(H^s, H^t)} \lesssim (1 + |\lambda|)^{t-s-1} e^{L(\Im \lambda)_-} \quad (3.2)$$

where $s \in \mathbb{R}$ and $t \in [s, s+2]$, $v_- := \max\{-v, 0\}$ and L is a constant larger than the diameter of the support of χ .

Proof. With the help of the functional calculus [4, Appendix B.2], we can denote

$$U(t) := \frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}},$$

which gives a representation of $\tilde{R}_0$ as follows

$$\tilde{R}_0(\lambda) = \int_0^\infty e^{i\lambda t} U(t) dt.$$

The spectral representation [4, Appendix B.1] directly gives

$$\partial_t^k U(t) : H^s(\mathbb{R}^3) \rightarrow H^{s-k+1}(\mathbb{R}^3), \quad k \in \mathbb{N}, \quad s \in \mathbb{R}.$$

Therefore, when $\Im \lambda > 0$, $\tilde{R}_0(\lambda) : H^s(\mathbb{R}^3) \rightarrow H^s(\mathbb{R}^3)$ is analytic with respect to λ . For any $f \in H^s$, denote $u(\lambda) = \tilde{R}_0(\lambda)f$, which yields $\Delta u(\lambda) = -\lambda^2 u(\lambda) - f \in H^s$. Hence we have both $u(\lambda)$ and $\Delta u(\lambda)$ are families of analytic functions in H^s when $\Im \lambda > 0$. For any $\phi \in H^{s+2}$, when $\Im \lambda > 0$, it is shown that

$$\langle u(\lambda), \phi \rangle_{H^{s+2}} = \langle (I - \Delta)u(\lambda), (I - \Delta)\phi \rangle_{H^s} = \langle \Delta u(\lambda), \Delta \phi \rangle_{H^s} + \langle u(\lambda), \phi \rangle_{H^s} - 2\langle \Delta u(\lambda), \phi \rangle_{H^s}$$

is analytic, which implies $u(\lambda)$ is a family of analytic functions in H^{s+2} . Consequently, we know that $\tilde{R}_0(\lambda)$ is a family of operators when $\Im \lambda > 0$ as follows

$$\tilde{R}_0(\lambda) : H^s \rightarrow H^{s+2}.$$

Next, we show the resolvent estimates and the analyticity of $\chi \tilde{R}_0(\lambda) \chi$ for $\lambda \in \mathbb{C}$. The strong Huygens's principle implies that

$$(U(t)\chi)(x) = 0, \quad t > \sup\{|x - y| : y \in \text{supp}\chi\}.$$

Thus we obtain

$$\chi \tilde{R}_0(\lambda) \chi = \int_0^L e^{i\lambda t} \chi U(t) \chi \, dt,$$

where $L > \text{diam supp}\chi$. Furthermore, we have

$$\chi \tilde{R}_0(\lambda) \chi = (i\lambda)^{-1} \int_0^L \partial_t (e^{i\lambda t}) \chi U(t) \chi \, dt = -(i\lambda)^{-1} \int_0^L e^{i\lambda t} \chi \partial_t (U(t)) \chi \, dt.$$

Applying the spectral representation yields $U(t) = \mathcal{O}_{\mathcal{L}(H^s, H^{s+1})}(1)$, $\partial_t U(t) = \cos t \sqrt{-\Delta} = \mathcal{O}_{\mathcal{L}(H^s, H^s)}(1)$. Hence we know when $s = t$ or $s = t + 1$,

$$\chi \tilde{R}_0(\lambda) \chi : H^s \rightarrow H^t,$$

is a family of analytic operators for $\lambda \in \mathbb{C}$ and the resolvent estimates (3.2) hold. Notice

$$\Delta(\chi \tilde{R}_0(\lambda) \chi) = \chi \Delta \tilde{R}_0(\lambda) \chi + [\Delta, \chi] \tilde{R}_0(\lambda) \chi = (-1 - \lambda^2) \chi \tilde{R}_0(\lambda) \chi + [\Delta, \chi] \tilde{R}_0(\lambda) \chi,$$

where $[\Delta, \chi] := \Delta \chi - \chi \Delta$ is a one order differential operator. Then the above identities and the resolvent estimates for $t = s, s + 1$ imply the estimate for $t = s + 2$. The remaining cases follow by applying the interpolation inequality. $\square$

As a corollary of the above lemma, we have the following resolvent estimates for the free biharmonic operator.

Lemma 3.2. *The free resolvent $R_0(\lambda)$ is analytic for $\lambda \in \mathbb{C}$ with $\Im \lambda > 0$, $\Re \lambda > 0$ as a family of operators*

$$R_0(\lambda) : H^s(\mathbb{R}^3) \rightarrow H^t(\mathbb{R}^3),$$

for all $s \in \mathbb{R}$ and $t \in [s, s + 4]$. Furthermore, given a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$, $R_0(\lambda)$ can be extended to a family of analytic operators for $\lambda \in \mathbb{C} \setminus \{0\}$ as follows

$$\chi R_0(\lambda) \chi : H^s(\mathbb{R}^3) \rightarrow H^t(\mathbb{R}^3),$$

with the resolvent estimates

$$\|\chi R_0(\lambda)\chi\|_{\mathcal{L}(H^s, H^t)} \lesssim (1 + |\lambda|)^{t-s-3} (e^{L(\Im\lambda)-} + e^{L(\Re\lambda)-}),$$

where $s \in \mathbb{R}$ and $t \in [s, s+4]$, $v_- := \max\{-v, 0\}$ and L is a constant larger than the diameter of the support of χ .

Lemma 3.3. Assume that $V \in W_{comp}^{s,p}(\mathbb{R}^3)$ with $s \in [-\frac{1}{2}, 0)$ and $p \in [-\frac{3}{s}, +\infty)$. We have

$$\|Vf\|_{H^s(\mathbb{R}^3)} \lesssim \|V\|_{W^{s,p}(\mathbb{R}^3)} \|\chi f\|_{H^{-s}(\mathbb{R}^3)}, \quad \forall f \in H_{loc}^{-s}(\mathbb{R}^3),$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is a cut-off function such that $\chi|_{\text{supp } V} = 1$.

Using the above lemmas, we can derive the following meromorphic continuation of the resolvent $R_{A,V}(\lambda) := (\Delta^2 + A \cdot \nabla + V - \lambda^4)^{-1}$.

Lemma 3.4. Assume $A \in W_{comp}^{s,p}(\mathbb{R}^3)^3$, $V \in W_{comp}^{s,p}(\mathbb{R}^3)$ with $s \in [-\frac{1}{2}, 0)$ and $p \in [-\frac{3}{s}, +\infty)$. Then the resolvent $R_{A,V}(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$, $\Re\lambda > 0$ as a family of operators $R_{A,V}(\lambda) : H^s \rightarrow H^t$ with $t \in [s, s+4]$, which can be extended to a family of meromorphic operators for

$$R_{A,V}(\lambda) := H_{comp}^s \rightarrow H_{loc}^t, \quad \lambda \in \mathbb{C}$$

with $t \in [s, s+4]$. Further, fix a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$. Then the resolvent $\chi R_{A,V}(\lambda) \chi : H^s \rightarrow H^s$ is analytic for $\lambda \in \mathcal{S}$ where

$$\mathcal{S} := \{\lambda \in \mathbb{C} : \Im\lambda \geq -A - \delta \log(1 + |\lambda|), \Re\lambda \geq -A - \delta \log(1 + |\lambda|), |\lambda| > C_0\}.$$

Moreover, there exist constants A, C_0, δ, C and L such that the resolvent estimates

$$\|\chi R_{A,V}(\lambda) \chi\|_{H^s \rightarrow H^t} \leq C(1 + |\lambda|)^{t-s-2} (e^{L(\Im\lambda)-} + e^{L(\Re\lambda)-})$$

hold for $t \in [s, s+4]$ and $\lambda \in \mathcal{S}$. The constants A, δ, L depend on χ and $\text{supp } V$. The constants C, C_0 depend on χ , $\|V\|_{W^{s,p}(\mathbb{R}^3)}$ and $\|A\|_{W^{s,p}(\mathbb{R}^3)^3}$.

Proof. Formally, we have the resolvent identity

$$R_{A,V}(\lambda) = R_0(\lambda)(I + (A \cdot \nabla + V)R_0(\lambda))^{-1}.$$

Similar as the proof of resolvent estimates for Lemma 3.2, it can be verified when $\Im\lambda > 0$, $\Re\lambda > 0$,

$$\begin{aligned} \|(A \cdot \nabla + V)R_0(\lambda)\|_{\mathcal{L}(H^s, H^s)} &\lesssim \|V\|_{W^{s,p}} \|R_0(\lambda)\|_{\mathcal{L}(H^s, H^{-s})} + \|A\|_{W^{s,p}} \|R_0(\lambda)\|_{\mathcal{L}(H^s, H^{-s+1})} \\ &\lesssim (1 + |\lambda|)^{-2-2s}. \end{aligned}$$

Hence, when $\Im\lambda > 0$, $\Re\lambda > 0$ and $|\lambda| \gg 1$, the operator $I + (A \cdot \nabla + V)R_0(\lambda) : H^s \rightarrow H^s$ is invertible by the following Neumann series argument

$$(I + (A \cdot \nabla + V)R_0(\lambda))^{-1} = \sum_{j=0}^{\infty} (-(A \cdot \nabla + V)R_0(\lambda))^j.$$

Moreover, from Lemma 3.2–3.3 we have that $(A \cdot \nabla + V)R_0(\lambda) : H^s \rightarrow H^s$ is a compact operator. Therefore, by applying the analytic Fredholm theory [4, Theorem C.8] we have that the resolvent

$$R_{A,V}(\lambda) = R_0(\lambda)(I + (A \cdot \nabla + V)R_0(\lambda))^{-1} : H^s \rightarrow H^t$$

is meromorphic with respect to λ with $\Im \lambda > 0$ and $\Re \lambda > 0$ for $t \in [s, s+4]$.

Next we consider the operator $\chi R_0(\lambda)\chi$. There holds the resolvent identity

$$\chi R_{A,V}(\lambda)\chi = \chi R_0(\lambda)\chi_1(I + (A \cdot \nabla + V)R_0(\lambda)\chi_1)^{-1}(1 - (A \cdot \nabla + V)R_0(\lambda)(1 - \chi_1))\chi, \quad (3.3)$$

where $\chi_1 \in C_0^\infty(\mathbb{R}^3)$ such that $\chi_1 = 1$ on $\text{supp } A \cup \text{supp } V$. By applying Lemma 3.2–3.3 and the analytic Fredholm theory again, we obtain that $R_{A,V} : H_{comp}^s \rightarrow H_{loc}^t$ can be extended to a family of meromorphic operators for $\lambda \in \mathbb{C}$ with $t \in [s, s+4]$. From Lemma 2.1 and Lemma 3.2–3.3 we have

$$\begin{aligned} \|A \cdot \nabla R_0(\lambda)\chi_1\|_{\mathcal{L}(H^s, H^s)} &\lesssim \|A\|_{W^{s,p}(\mathbb{R}^3)} \|\chi R_0(\lambda)\chi\|_{\mathcal{L}(H^s, H^{-s+1})} \\ &\lesssim (1 + |\lambda|)^{-2-2s} (e^{L(\Im \lambda)-} + e^{L(\Re \lambda)-}), \quad \lambda \in \mathcal{S}, \end{aligned}$$

and similarly

$$\|V R_0(\lambda)\chi_1\|_{\mathcal{L}(H^s, H^s)} \lesssim \|V\|_{W^{s,p}(\mathbb{R}^3)} (1 + |\lambda|)^{-3-2s} (e^{L(\Im \lambda)-} + e^{L(\Re \lambda)-}), \quad \lambda \in \mathcal{S}.$$

Consequently, there holds

$$\|(A \cdot \nabla + V)R_0(\lambda)\chi_1\|_{\mathcal{L}(H^s, H^s)} \lesssim (1 + |\lambda|)^{-2-2s} (e^{L(\Im \lambda)-} + e^{L(\Re \lambda)-}), \quad \lambda \in \mathcal{S}. \quad (3.4)$$

For sufficiently small $\delta < \frac{1}{2L}$ and sufficiently large C_0 , we have

$$\|(A \cdot \nabla + V)R_0(\lambda)\chi\|_{\mathcal{L}(H^s, H^s)} \lesssim |\lambda|^{-1/2} < 1.$$

Thus, from a Neumann series argument we have $(I + (A \cdot \nabla + V)R_0(\lambda)\chi)^{-1} : H_{comp}^s \rightarrow H_{comp}^s$ is invertible. From the resolvent identity (3.3), the free resolvent estimates in Lemma 3.2 and the analytic Fredholm theory, we have that $\chi R_{A,V}(\lambda)\chi : H^s \rightarrow H^t$ is analytic for $\lambda \in \mathcal{S}$ with the resolvent estimate

$$\|\chi R_{A,V}(\lambda)\chi\|_{\mathcal{L}(H^s, H^t)} \lesssim (1 + |\lambda|)^{t-s-2} (e^{L(\Im \lambda)-} + e^{L(\Re \lambda)-}), \quad t \in [s, s+4],$$

which completes the proof. $\square$

Based on the above arguments, we establish the well-posedness of the direct scattering problem.

Theorem 3.1. Assume that the random potential A is a GMIG random field of order $-m_A$ in $D \subset \mathbb{R}^3$ and V is a GMIG random field of order $-m_V$ in D with $m_A > 2$, $m_V > 2$. For k sufficiently large, the scattering problem (1.1) almost surely admits a unique solution $u^{sc} \in H_{loc}^{s+4}$ with non-positive constant s satisfying $-1/2 < s < \min\{(m_A - 3)/2, (m_V - 3)/2\}$.

Proof. Applying Lemma 2.1 gives $A \in W^{s,p}(D)^3$ and $V \in W^{s,p}(D)$ with non-positive s satisfying $-1/2 < s < \min\{(m_A - 3)/2, (m_V - 3)/2\}$ and $p \in [-\frac{3}{s}, +\infty)$. In (3.3), taking χ and χ_1 such that $\chi = 1$ in D and $\chi_1 = 1$ in $\text{supp } \chi$, we can rewrite (3.3) as

$$R_{A,V}(k)\chi = R_0(k)\chi_1(I + (A \cdot \nabla + V)R_0(k)\chi_1)^{-1}\chi. \quad (3.5)$$

Then combining (3.5) and Lemma 3.3-3.4 yields for sufficiently large $k > 0$ that

$$u^{sc} := R_{A,V}(k)((-A \cdot \nabla + V)u^{inc}) = R_{A,V}(k)\chi((-A \cdot \nabla + V)u^{inc})$$

is well defined and $u^{sc} \in H_{loc}^{s+4}$, which is a solution to the scattering problem (1.1).

Next we proceed to prove the uniqueness. Denote the fundamental solution of the biharmonic equation by

$$G(x, y, k) = \frac{1}{2k^2} \left(\frac{e^{ik|x-y|}}{4\pi|x-y|} - \frac{e^{-k|x-y|}}{4\pi|x-y|} \right)$$

which is the kernel of $R_0(\lambda)$. Assume $\tilde{u}^{sc} \in H_{loc}^{s+4}$ is a solution to the direct scattering problem (1.1) in distribution sense. Next we show that $\tilde{u}^{sc}$ satisfies the Lippmann-Schwinger equation

$$\tilde{u}^{sc}(x) = - \int_{\mathbb{R}^3} G(x, y, k)(A(y) \cdot \nabla + V(y))(e^{ikd \cdot y} + \tilde{u}^{sc}(y)) dy. \quad (3.6)$$

To this end, choose a sequence $\{\phi_n\}_{n=1}^\infty \subset C^\infty(\mathbb{R}^3)$ such that $\phi_n \rightarrow \tilde{u}^{sc}$ in H_{loc}^{s+4} . Then using integration by parts gives

$$\begin{aligned} & \int_{B_R} (\Delta^2 - k^4)\phi_n(y)G(x, y, k) dy \\ &= \phi_n(x) + \int_{\partial B_R} (\partial_{\nu(y)}(\Delta_y \phi_n(y))G(x, y, k) - \Delta \phi_n(y)\partial_{\nu(y)}G(x, y, k)) ds \\ &+ \int_{\partial B_R} (\partial_{\nu(y)}\phi_n(y)\Delta_y G(x, y, k) - \phi_n(y)\partial_{\nu(y)}(\Delta_y G(x, y, k))) ds \quad \text{for } x \in B_R. \end{aligned} \quad (3.7)$$

From Sobolev embedding theorem we have $\tilde{u}^{sc} \in H^{s+4}(B_R) \subset C(B_R)$, which implies $\phi_n(x) \rightarrow \tilde{u}^{sc}(x)$. Noticing $(\Delta^2 - k^4)\tilde{u}^{sc} \in H_{comp}^s(B_R)$, by applying Lemma 3.2 in [15] and Sobolev embedding theorem we obtain $G(x, \cdot, k) \in H^{-s}(B_R)$ as $s > -\frac{1}{2}$, which gives

$$\lim_{n \rightarrow \infty} \int_{B_R} (\Delta^2 - k^4) \phi_n(y) G(x, y, k) dy = \int_{B_R} (\Delta^2 - k^4) \tilde{u}^{sc}(y) G(x, y, k) dy.$$

Therefore, by letting $n \rightarrow \infty$ in (3.7) we arrive at

$$\begin{aligned} \tilde{u}^{sc}(x) &= \int_{B_R} (\Delta^2 - k^4) \tilde{u}^{sc}(y) G(x, y, k) dy \\ &+ \int_{\partial B_R} \left(\partial_{\nu(y)} (\Delta_y \tilde{u}^{sc}(y)) G(x, y, k) - \Delta \tilde{u}^{sc}(y) \partial_{\nu(y)} G(x, y, k) \right) ds \\ &+ \int_{\partial B_R} \left(\partial_{\nu(y)} \tilde{u}^{sc}(y) \Delta_y G(x, y, k) - \tilde{u}^{sc}(y) \partial_{\nu(y)} (\Delta_y G(x, y, k)) \right) ds, \quad x \in B_R, \end{aligned}$$

which yields the Lippmann-Schwinger equation (3.6) by taking $R \rightarrow \infty$ and using the radiation condition.

At last, we show that when k is sufficient large, the Lippmann-Schwinger equation admits a unique solution, which ensures the uniqueness of the scattering problem. It suffices to verify that the homogeneous equation

$$v(x) = - \int_{\mathbb{R}^3} G(x, y, k) (A(y) \cdot \nabla + V(y)) v(y) dy$$

only admits the trivial solution in H_{loc}^{-s} . The above equation can be rewritten as

$$v = R_0(k) \chi (A \cdot \nabla + V) v.$$

Then applying Lemma 3.2–3.3 gives

$$\|\chi v\|_{H^{-s}} \leq \frac{C(\chi)}{k^{2s+2}} \|\chi v\|_{H^{-s}},$$

which implies $\chi v \equiv 0$ as k is large enough. Thus, noting $\chi = 1$ in D , we have $v = 0$ in D . Since v satisfies the Helmholtz equation

$$(\Delta^2 - k^4)v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

and $\text{supp } A, \text{supp } V \subset\subset D$, the unique continuation for elliptic equations implies $v \equiv 0$ in $\mathbb{R}^3$. The proof is complete. $\square$

Based on the above analysis, the scattered field u^{sc} admits a Born series expansion, which is useful for the subsequent study of the inverse problem. Indeed, we have

$$\begin{aligned}
u^{sc} &:= R_{A,V}(k)((-A \cdot \nabla + V)u^{inc}) = R_{A,V}(k)\chi((-A \cdot \nabla + V)u^{inc}) \\
&= R_0(k)\chi_1(I + (A \cdot \nabla + V)R_0(k)\chi_1)^{-1}\chi((-A \cdot \nabla + V)u^{inc}) \\
&= -R_0(k)\sum_{j=0}^{\infty}(-A \cdot \nabla + V)R_0(k))^j((A \cdot \nabla + V)u^{inc}), \tag{3.8}
\end{aligned}$$

and from Lemma 3.2–3.3 the series

$$\sum_{j=0}^{\infty}(-A \cdot \nabla + V)R_0(k))^j((A \cdot \nabla + V)u^{inc}) = \sum_{j=0}^{\infty}(-A \cdot \nabla + V)\chi R_0(k)\chi)^j((A \cdot \nabla + V)u^{inc})$$

converges in H^s for sufficiently large k . In Theorem 3.1, we obtain the well-posedness of the direct scattering problem (1.1) only for the large wavenumber. In fact, the condition that k is sufficiently large is used to ensure that the solution admits the above Born series expansion. This condition may not be necessary to prove the well-posedness of the direct scattering problem (1.1). Similarly as [9], we may use the unique continuation principle and the Rellich lemma to get the uniqueness, and then apply the Fredholm alternative to obtain the existence of the direct scattering problem (1.1). In this paper we only consider the large wavenumber case since for the inverse problem we need to use the Born series expansion of the solution.

4. Inverse scattering problem

In this section, we investigate the inverse scattering problem of recovering the micro-correlation strengths $\mathbb{H}$ and h from the far-field pattern. The scattered wave u^{sc} admits the asymptotic expansion as follows

$$u^{sc}(x) = \frac{e^{ik|x|}}{|x|}u^{\infty}(\hat{x}, d, k) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \rightarrow \infty,$$

where $u^{\infty}(\hat{x}, d, k)$ is referred to as the far-field pattern with $\hat{x} = \frac{x}{|x|}$. Now we expand the far-field pattern into Born series. For sufficiently large k , from (3.8) we have the Born series expansion for the scattered field

$$u^{sc} = -R_0(k)\sum_{j=0}^{\infty}(-A \cdot \nabla + V)R_0(k))^j((A \cdot \nabla + V)u^{inc}).$$

Notice that for fixed y and large $|x|$, there holds

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x} \cdot y}(1 + \mathcal{O}(|x|^{-1})), \quad |x| \rightarrow \infty,$$

and $\frac{e^{-k|x-y|}}{|x-y|}$ decays exponentially. Consequently, we have

$$G(x, y, k) = \frac{1}{8\pi k^2}e^{-ik\hat{x} \cdot y} \frac{e^{ik|x|}}{|x|}(1 + \mathcal{O}(|x|^{-1})), \quad |x| \rightarrow \infty,$$

and the far-field pattern can be represented by Born series

$$\begin{aligned}
 u^\infty(\hat{x}, d, k) &= -\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} (A(y) \cdot \nabla + V(y)) \sum_{j \geq 0} [(-R_0(k)(A \cdot \nabla + V(y)))^j u^{inc}](y) dy \\
 &= -\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} [(A(y) \cdot \nabla + V(y)) u^{inc}](y) dy \\
 &\quad - \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} (A(y) \cdot \nabla + V(y)) [(-R_0(k)(A \cdot \nabla + V(y))) u^{inc}](y) dy \\
 &\quad - \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} (A(y) \cdot \nabla + V(y)) \sum_{j \geq 2} [(-R_0(k)(A \cdot \nabla + V(y)))^j u^{inc}](y) dy \\
 &:= u_0^\infty(\hat{x}; \theta, k) + u_1^\infty(\hat{x}; \theta, k) + u_2^\infty(\hat{x}; \theta, k).
 \end{aligned}$$

Now we are in the position to state main uniqueness results in this paper. Next use $\langle \hat{x}, d \rangle$ to denote the angle between the two unit vectors $\hat{x}, d$.

Theorem 4.1. Assume $2 < m_A < 4$, $m_V > 2$ and $m_A < 6m_V - 10$. Moreover, suppose that A and V are independent GMIG random fields. Let $\hat{x}, d, \hat{x}', d' \in \mathbb{S}^2$ be such that $\hat{x} - d = \hat{x}' - d'$ and the unit vectors $e_1 = \frac{d+\hat{x}}{|d+\hat{x}|}$, $e_2 = \frac{\hat{x}-d}{|\hat{x}-d|}$ and $e_3 = \frac{\hat{x}'+d'}{|\hat{x}'+d'|}$ are mutually orthogonal. Then it holds almost surely that

$$\widehat{\mathbb{H}}(\tau(\hat{x} - d)) = -2^{\frac{m_A}{2}+5} \pi^2 \lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} \sum_{i,j=1}^3 k^{m_A+2} (J_i(\hat{x}, d, k + \tau) \overline{J_j(\hat{x}, d, k)}) e_i e_j^T dk \quad (4.1)$$

for all $\tau > 0$ and all $(\hat{x}, d)$ in

$$\{(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \hat{x}, d \rangle = \pi/2, \hat{x} - d \notin S_0\}.$$

Here $S_0 \subset \sqrt{2}\mathbb{S}^2$ is a set of measure zero and $J_i, i = 1, 2, 3$, are given by

$$\begin{aligned}
 J_1(\hat{x}, d, k) &= u^\infty(\hat{x}, d, k) - \overline{u^\infty(d, \hat{x}, k)}, \\
 J_2(\hat{x}, d, k) &= -(u^\infty(\hat{x}, d, k) + \overline{u^\infty(d, \hat{x}, k)}), \\
 J_3(\hat{x}, d, k) &= u^\infty(\hat{x}', d', k) - \overline{u^\infty(d', \hat{x}', k)}.
 \end{aligned}$$

Remark 4.1.

- (1) For almost every $\xi \in \mathbb{R}^3$, there exists $\tau > 0$ and $\hat{x}, d \in \mathbb{S}^2$ such that $\langle \hat{x}, d \rangle = \pi/2$, $\hat{x} - d \notin S_0$ and $\xi = \tau(\hat{x} - d)$. Consequently, the micro-correlation strength is uniquely determined from the limit (4.1).
- (2) It is not essential that we choose $\langle \hat{x}, d \rangle = \pi/2$. In fact we can choose any $\alpha \in (0, \pi)$ and then let $\langle \hat{x}, d \rangle = \alpha$. Similar uniqueness results as (4.1) will also hold.

Theorem 4.2. *Let $1 + m_V < m_A$, $\frac{12}{5} < m_V \leq 5$ and $m_V < 6m_A - 12$. Moreover, suppose that A and V are independent GMIG random fields. Then for all $\tau > 0$ and almost every $d \in \mathbb{S}^2$ it holds almost surely that*

$$\begin{aligned} & \widehat{h}(\tau(\hat{x} - d)) \\ &= \lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} 64\pi^2 k^{m_V+4} u^\infty(-d, d, k + \tau) \overline{u^\infty(-d, d, k)} + \sum_{i,j=1}^3 k^{2+m_V-m_A} d_i d_j \widehat{\mathbb{H}}_{ij}(-2\tau\hat{x}) dk. \end{aligned} \quad (4.2)$$

The limit in (4.2) shows if $\mathbb{H}$ is known, then h can be uniquely determined from a single realization of the correlation backscattering far-field data by letting $\hat{x} = -d$ almost surely. As a corollary of Theorem 4.1 and Theorem 4.2, we have the following uniqueness of determining both $\mathbb{H}$ and h by applying the analyticity of the resolvent in Lemma 3.4.

Theorem 4.3. *Let $1 + m_V < m_A < 4$ and $m_V > \frac{12}{5}$. Then both $\mathbb{H}$ and h can be uniquely determined from the far-field pattern $\{u^\infty(\hat{x}, d, k) : k \in I, (\hat{x}, d) \in \mathbb{B}\}$ where*

$$\mathbb{B} = \{(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \hat{x}, d \rangle = \pi/2, \hat{x} - d \notin S_0\} \cup \{(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2 : \hat{x} = -d\}.$$

Here $S_0 \subset \sqrt{2}\mathbb{S}^2$ is a set of measure zero and $I = [K_0, K_1]$ where $K_1 > K_0 > C_0$ with C_0 being specified in Lemma 3.4.

In next two subsections, we will prove Theorem 4.1 and Theorem 4.2, respectively. Before that, we build the relation between the Fourier transform of the micro-correlation strength and the Fourier transform of the random potential. We have

$$\begin{aligned} \mathbb{E}[\widehat{A}(\xi) \overline{\widehat{A}(\eta)}^T] &= \mathbb{E} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A(x) A(y)^T e^{-i\xi \cdot x} e^{i\eta \cdot y} dx dy \right) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_A(x, y) e^{-i\xi \cdot x} e^{i\eta \cdot y} dx dy \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-i(\xi \cdot x - \eta \cdot y)} e^{i(x-y) \cdot \zeta} c_A(x, \zeta) d\zeta dx dy \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix \cdot (\zeta - \xi)} c_A(x, \zeta) \delta(\eta - \zeta) d\zeta dx = \int_D e^{-ix \cdot (\xi - \eta)} c_A(x, \eta) dx \\ &= \int_{\mathbb{R}^3} e^{-ix \cdot (\xi - \eta)} \mathbb{H}(x) |\eta|^{-m_A} dx + O(|\eta|^{-(m_A+1)}) \\ &= |\eta|^{-m_A} \widehat{\mathbb{H}}(\xi - \eta) + \mathcal{O}(|\eta|^{-(m_A+1)}), \quad |\eta| \rightarrow \infty. \end{aligned} \quad (4.3)$$

Similarly, there holds

$$\mathbb{E}[\widehat{V}(\xi)\overline{\widehat{V}(\eta)}] = |\eta|^{-m_V}\widehat{h}(\xi - \eta) + O(|\eta|^{-(m_V+1)}), \quad |\eta| \rightarrow \infty. \quad (4.4)$$

In order to determine $\mathbb{H}$ and h , we will establish relations between the left hand sides of (4.3)–(4.4) and the correlation of the far-field pattern.

4.1. Recovery of the first order perturbation

In this section, we will prove Theorem 4.1. We firstly show the determining of $\mathbb{H}$ from the zeroth-order term of the far-field pattern.

Lemma 4.1. *Given $\hat{x}, d \in \mathbb{S}^2$ with $\langle \hat{x}, d \rangle = \frac{\pi}{2}$, take $\hat{x}', d' \in \mathbb{S}^2$ such that $\hat{x} - d = \hat{x}' - d'$. If $m_A < 2 + m_V$, then there holds the limit*

$$\widehat{\mathbb{H}}(\tau(\hat{x} - d)) = -2^{\frac{m_A}{2}+5}\pi^2 \lim_{k \rightarrow \infty} k^{m_A+2} \sum_{i,j=1}^3 \mathbb{E}[I_i(\hat{x}, d, k + \tau) \overline{I_j(\hat{x}, d, k)}] e_i e_j^T$$

for all $\tau > 0$, where e_1, e_2, e_3 are mutually orthogonal unit vectors given as $e_1 = \frac{d+\hat{x}}{|d+\hat{x}|}$, $e_2 = \frac{\hat{x}-d}{|\hat{x}-d|}$, $e_3 = \frac{\hat{x}'+d'}{|\hat{x}'+d'|}$ and $I_j(\hat{x}, d, k)$ are given by

$$\begin{aligned} I_1(\hat{x}, d, k) &= u_0^\infty(\hat{x}, d, k) - \overline{u_0^\infty(d, \hat{x}, k)}, \\ I_2(\hat{x}, d, k) &= -(u_0^\infty(\hat{x}, d, k) + \overline{u_0^\infty(d, \hat{x}, k)}), \\ I_3(\hat{x}, d, k) &= u_0^\infty(\hat{x}', d', k) - \overline{u_0^\infty(d', \hat{x}', k)}. \end{aligned}$$

Proof. The zeroth-order term of the far-field pattern can be represented by

$$u_0^\infty(\hat{x}, d, k) = -\frac{1}{8\pi} \left(\frac{i}{k} \int_{\mathbb{R}^3} (\mathbf{A}(\mathbf{y}) \cdot d) e^{-iky \cdot (\hat{x}-d)} d\mathbf{y} + \frac{1}{k^2} \int_{\mathbb{R}^3} V(\mathbf{y}) e^{-iky \cdot (\hat{x}-d)} d\mathbf{y} \right). \quad (4.5)$$

We use the far-field pattern to express the Fourier transform of $\mathbf{A}$. To this end, notice

$$\begin{aligned} \widehat{\mathbf{A}}(k(\hat{x} - d)) &= \sum_{j=1}^3 (\widehat{\mathbf{A}}(k(\hat{x} - d)) \cdot e_j) e_j = e_1 \int_{\mathbb{R}^3} \mathbf{A}(\mathbf{y}) \cdot \frac{\hat{x} + d}{|\hat{x} + d|} e^{-iky \cdot (\hat{x}-d)} d\mathbf{y} \\ &\quad + e_2 \int_{\mathbb{R}^3} \mathbf{A}(\mathbf{y}) \cdot \frac{\hat{x} - d}{|\hat{x} - d|} e^{-iky \cdot (\hat{x}-d)} d\mathbf{y} + e_3 \int_{\mathbb{R}^3} \mathbf{A}(\mathbf{y}) \cdot \frac{\hat{x}' + d'}{|\hat{x}' + d'|} e^{-iky \cdot (\hat{x}-d)} d\mathbf{y}, \end{aligned} \quad (4.6)$$

Combining (4.5)–(4.6) and direct calculations give

$$\begin{aligned} \widehat{\mathbf{A}}(k(\hat{x} - d)) + \frac{\sqrt{2}\mathbf{i}}{k} \widehat{\mathbf{V}}(k(\hat{x} - d))e_2 &= 4\sqrt{2}\pi k \mathbf{i} (u_0^\infty(\hat{x}, d, k) - \overline{u_0^\infty(d, \hat{x}, k)})e_1 \\ &\quad - 4\sqrt{2}\pi k \mathbf{i} (u_0^\infty(\hat{x}, d, k) + \overline{u_0^\infty(d, \hat{x}, k)})e_2 + 4\sqrt{2}\pi k \mathbf{i} (u_0^\infty(\hat{x}', d', k) - \overline{u_0^\infty(d', \hat{x}', k)})e_3 \\ &:= 4\sqrt{2}\pi k \mathbf{i} \sum_{j=1}^3 I_j(\hat{x}, d, k)e_j. \end{aligned}$$

By taking covariance we have

$$\begin{aligned} \mathbb{E}[\widehat{\mathbf{A}}((k + \tau)(\hat{x} - d))\overline{\widehat{\mathbf{A}}(k(\hat{x} - d))}^T] &- \frac{2}{k^2} \mathbb{E}[\widehat{\mathbf{V}}((k + \tau)(\hat{x} - d))\overline{\widehat{\mathbf{V}}(k(\hat{x} - d))}]e_2e_2^T \\ &= -2^{\frac{m_A}{2}+5}\pi^2k^2 \sum_{i,j=1}^3 \mathbb{E}[I_i(\hat{x}, d, k + \tau)\overline{I_j(\hat{x}, d, k)}]e_ie_j^T \end{aligned}$$

Letting $\xi = (k + \tau)(\hat{x} - d)$ and $\eta = k(\hat{x} - d)$ in (4.3), we obtain

$$\mathbb{E}[\widehat{\mathbf{A}}((k + \tau)(\hat{x} - d))\overline{\widehat{\mathbf{A}}(k(\hat{x} - d))}^T] = (\sqrt{2}k)^{-m_A} \widehat{\mathbb{H}}(\tau(\hat{x} - d)) + O(|k|^{-(m_A+1)}), \quad k \rightarrow \infty. \quad (4.7)$$

Combining (4.3), (4.4) and (4.7), we have

$$\begin{aligned} \widehat{\mathbb{H}}(\tau(\hat{x} - d)) &= -2^{\frac{m_A}{2}+5}\pi^2k^{m_A+2} \sum_{i,j=1}^3 \mathbb{E}[I_i(\hat{x}, d, k + \tau)\overline{I_j(\hat{x}, d, k)}]e_ie_j^T \\ &\quad + O(k^{-1}) + O(k^{m_A-2-m_V}). \quad \square \end{aligned}$$

The results in Lemma 4.1 imply that $\mathbb{H}$ can be uniquely determined by the expectation of high frequency limit of the far-field data. However, the expectation requires the measurements at all sample paths of the random potentials. The following lemma shows that $\mathbb{H}$ can also be recovered by the far-field data at a single sample almost surely.

Lemma 4.2. *For $2 < m_A < m_V + 2$, $m_V > 2$, there holds*

$$\widehat{\mathbb{H}}(\tau(\hat{x} - d)) = -2^{\frac{m_A}{2}+5}\pi^2 \lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} \sum_{i,j=1}^3 k^{m_A+2} (I_i(\hat{x}, d, k + \tau)\overline{I_j(\hat{x}, d, k)})e_ie_j^T dk$$

for all $\tau > 0$ and $\hat{x}, d \in \mathbb{S}^2$ with $\langle \hat{x}, d \rangle = \frac{\pi}{2}$.

In order to prove the above lemma, one only needs to show that

$$\begin{aligned} \lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} (I_i(\hat{x}, d, k + \tau)\overline{I_j(\hat{x}, d, k)} - \mathbb{E}[I_i(\hat{x}, d, k + \tau)\overline{I_j(\hat{x}, d, k)}]) dk &= 0, \\ i, j &= 1, 2, 3. \end{aligned} \quad (4.8)$$

To this end, we need the following lemma on ergodicity of the stochastic process [3, p. 91].

Lemma 4.3. *Let X_t with $t \geq 0$ be a real-valued stochastic process with continuous paths. In addition, the process X_t satisfies $\mathbb{E}X_t = 0$ and there exist non-negative constants c_0, c_1, c_2 , satisfying $0 \leq 2c_1 < c_2 < 1$ such that*

$$|\mathbb{E}(X_s X_t)| \lesssim \frac{s^{c_1} + t^{c_1}}{1 + |s - t|^{c_2}}$$

for any $s, t \geq c_0$. Then it holds almost surely that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{2T} X_t \, dt = 0.$$

Proof of Lemma 4.2. It is straightforward to verify that $\mathbb{E}[I_i(\hat{x}, d, k + \tau) \overline{I_j(\hat{x}, d, k)}]$ is a linear combination of the following terms

$$\begin{aligned} & \mathbb{E}[u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau)], \mathbb{E}[u_0^\infty(\hat{x}_1, d_1, k) \overline{u_0^\infty(\hat{x}_2, d_2, k + \tau)}], \\ & \mathbb{E}[u_0^\infty(\hat{x}_1, d_1, k + \tau) \overline{u_0^\infty(\hat{x}_2, d_2, k)}], \mathbb{E}[\overline{u_0^\infty(\hat{x}_1, d_1, k)} u_0^\infty(\hat{x}_2, d_2, k + \tau)] \end{aligned}$$

where

$$\hat{x}_j, d_j \in \{\hat{x}, d, \hat{x}', d'\} \quad \text{with} \quad \hat{x}_j - d_j = \pm(\hat{x} - d), \quad j = 1, 2.$$

Thus, in order to prove (4.8), it suffices to verify

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} (u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau) - \mathbb{E}[u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau)]) \, dk = 0, \quad (4.9)$$

since other terms can be treated in a similar way. By direct calculations, we obtain

$$\begin{aligned} & u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau) - \mathbb{E}[u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau)] \\ &= \sum_{W_k \in \mathcal{A}} C(W_k) (W_k^2 - \mathbb{E} W_k^2), \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} := & \left\{ \Re u_0^\infty(\hat{x}_1, d_1, k), \Im u_0^\infty(\hat{x}_1, d_1, k), \Re u_0^\infty(\hat{x}_2, d_2, k + \tau), \Im u_0^\infty(\hat{x}_2, d_2, k + \tau), \right. \\ & \Re u_0^\infty(\hat{x}_1, d_1, k) - \Re u_0^\infty(\hat{x}_2, d_2, k + \tau), \Im u_0^\infty(\hat{x}_1, d_1, k) + \Im u_0^\infty(\hat{x}_2, d_2, k + \tau), \\ & \left. \Re u_0^\infty(\hat{x}_1, d_1, k) - \Im u_0^\infty(\hat{x}_2, d_2, k + \tau), \Im u_0^\infty(\hat{x}_1, d_1, k) - \Re u_0^\infty(\hat{x}_2, d_2, k + \tau) \right\} \end{aligned}$$

and the constants $C(W_k) \in \{\frac{1+i}{2}, -\frac{1}{2}, -\frac{i}{2}\}$. Let $X_k := k^{m_A+2}(W_k^2 - \mathbb{E} W_k^2)$. Recalling u_0^∞ is given by (4.5) and $|\hat{x}_1 - d_1| = |\hat{x}_2 - d_2|$, according to [2, Proposition 4.3], we have

$$|\mathbb{E}(X_{k_1} X_{k_2})| \lesssim \frac{1 + k_1^{m_A - m_V - 2}}{1 + |k_1 - k_2|}$$

for $k_1, k_2 > 1$. Notice that $m_A - m_V - 2 < 1/2$. Then applying Lemma 4.3 yields (4.8). Combining Lemma 4.1 and (4.8) completes the proof. $\square$

Notice that for any $\xi \in \mathbb{R}^3$, there exists $\tau > 0$ and $\hat{x}, d \in \mathbb{S}^2$ such that $\langle \hat{x}, d \rangle = \frac{\pi}{2}$ and $\xi = \tau(\hat{x} - d)$. Therefore, the above lemma shows the unique determination of $\mathbb{H}$ from a single realization of zero-order terms in Born series of the far-field pattern.

In what follows, in order to determine $\mathbb{H}$ from the far-field pattern, we estimate the error between the correlation of the far-field pattern and that of the zeroth-order term in Born series. Denote

$$\begin{aligned} J_1(\hat{x}, d, k) &= u^\infty(\hat{x}, d, k) - \overline{u^\infty(d, \hat{x}, k)}, \\ J_2(\hat{x}, d, k) &= -(u^\infty(\hat{x}, d, k) + \overline{u^\infty(d, \hat{x}, k)}), \\ J_3(\hat{x}, d, k) &= u^\infty(\hat{x}', d', k) - \overline{u^\infty(d', \hat{x}', k)}. \end{aligned}$$

It is enough show

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} \sum_{i,j=1}^3 k^{m_A+2} (I_i(\hat{x}, d, k + \tau) \overline{I_j(\hat{x}, d, k)} - J_i(\hat{x}, d, k + \tau) \overline{J_j(\hat{x}, d, k)}) e_i e_j^T dk = 0. \quad (4.10)$$

Similar to the arguments above, we can derive (4.10) from

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} (u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau) - u^\infty(\hat{x}_1, d_1, k) u^\infty(\hat{x}_2, d_2, k + \tau)) dk = 0 \quad (4.11)$$

with

$$\hat{x}_j, d_j \in \{\hat{x}, d, \hat{x}', d'\} \quad \text{such that} \quad \hat{x}_j - d_j = \pm(\hat{x} - d), \quad j = 1, 2.$$

Notice that by applying the Cauchy-Schwartz inequality we have

$$\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_0^\infty(\hat{x}_1, d_1, k) u_0^\infty(\hat{x}_2, d_2, k + \tau) - u^\infty(\hat{x}_1, d_1, k) u^\infty(\hat{x}_2, d_2, k + \tau)| dk$$

$$\begin{aligned}
& \lesssim \sum_{i,j=1}^2 \frac{1}{K} \left(\int_K^{2K} k^{m_A+2} |u_i^\infty(\hat{x}_1, d_1, k)|^2 dk + \int_K^{2K} k^{m_A+2} |u_j^\infty(\hat{x}_2, d_2, k + \tau)|^2 dk \right) \\
& + \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_i^\infty(\hat{x}_2, d_2, k + \tau)|^2 dk} \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_0^\infty(\hat{x}_1, d_1, k)|^2 dk} \\
& + \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_j^\infty(\hat{x}_1, d_1, k)|^2 dk} \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_0^\infty(\hat{x}_2, d_2, k + \tau)|^2 dk} \\
& \lesssim \sum_{i,j=1}^2 \frac{1}{K} \left(\int_K^{2K} k^{m_A+2} |u_i^\infty(\hat{x}_1, d_1, k)|^2 dk + \int_K^{2K} k^{m_A+2} |u_j^\infty(\hat{x}_2, d_2, k + \tau)|^2 dk \right) \\
& + \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_i^\infty(\hat{x}_2, d_2, k + \tau)|^2 dk} + \sqrt{\frac{1}{K} \int_K^{2K} k^{m_A+2} |u_j^\infty(\hat{x}_1, d_1, k)|^2 dk}. \quad (4.12)
\end{aligned}$$

Hence, to prove (4.11), we give the following two lemmas, which show under some proper conditions, the right-hand side of (4.12) vanishes as K turns to infinity.

Lemma 4.4. *For $2 < m_A < 4$, there almost surely holds*

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} |u_1^\infty(\hat{x}, d, k)|^2 dk = 0$$

for all $(\hat{x}, d)$ in the set

$$\{(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \hat{x}, d \rangle = \pi/2, \hat{x} - d \notin S_0\},$$

where $S_0 \subset \sqrt{2}\mathbb{S}^2$ is a set of measure zero.

Lemma 4.5. *When $2 < m_A < 6m_V - 10$ and $m_A < 8$, there almost surely holds*

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} |u_2^\infty(\hat{x}, d, k)|^2 dk = 0$$

for all $\hat{x}, d \in \mathbb{S}^2$.

Proof of Lemma 4.4. By applying integration by parts we have

$$u_1^\infty(\hat{x}, d, k) = \frac{-1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} (A(y) \cdot \nabla + V(y)) [(-R_0(k)(A \cdot \nabla + V))u^{inc}](y) dy$$

$$\begin{aligned}
&= \frac{-1}{8\pi k^2} \int_{\mathbb{R}^3} (e^{-ik\hat{x}\cdot y} (-ik\hat{x}) \cdot \mathbf{A}(y) + e^{-ik\hat{x}\cdot y} V(y)) [(-R_0(k)(\mathbf{A} \cdot \nabla + V))u^{inc}](y) dy \\
&= \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d\cdot s - \hat{x}\cdot y)} G(y, s) V(y) V(s) ds dy \\
&\quad - \frac{i}{8\pi k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d\cdot s - \hat{x}\cdot y)} G(y, s) (\hat{x} \cdot \mathbf{A}(y)) V(s) ds dy \\
&\quad + \frac{i}{8\pi k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d\cdot s - \hat{x}\cdot y)} G(y, s) (d \cdot \mathbf{A}(s)) V(y) ds dy, \\
&\quad - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d\cdot s - \hat{x}\cdot y)} G(y, s) (\hat{x} \cdot \mathbf{A}(y)) (d \cdot \mathbf{A}(s)) ds dy := \frac{1}{8\pi} \sum_{j=1}^4 u_{1,j}^\infty(\hat{x}, d, k).
\end{aligned}$$

Subsequently, it remains to prove that

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} |u_{1,j}^\infty(\hat{x}, d, k)|^2 dk = 0, \quad j = 1, \dots, 4. \quad (4.13)$$

By the dominated convergence theorem, (4.13) is implied by

$$\int_1^\infty k^{m_A+1} |u_{1,j}^\infty(\hat{x}, d, k)|^2 dk < \infty, \quad j = 1, \dots, 4. \quad (4.14)$$

We only prove (4.14) for $u_{1,4}^\infty(\hat{x}, d, k)$ since under the assumption that $\mathbf{A}$ and V are independent, the other terms can be estimated similarly and they even decay faster when $k \rightarrow \infty$. To this end, consider the mollification $\mathbf{A}_\epsilon := \mathbf{A} * \chi_\epsilon$, where $\chi_\epsilon(x) = \frac{1}{\epsilon^3} \chi(\frac{x}{\epsilon})$ with $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} \chi dx = 1$. Then consider

$$\tilde{u}_{1,4,\epsilon}^\infty(\hat{x}, d, k) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d\cdot s - \hat{x}\cdot y)} G(y, s) (\hat{x} \cdot \mathbf{A}_\epsilon(y)) (d \cdot \mathbf{A}_\epsilon(s)) ds dy$$

Taking $z = s + y$, $z' = y - s$ and then letting $z' = \rho\omega$, we obtain

$$\begin{aligned}
u_{1,4,\epsilon}^\infty(\theta, \theta', k) &:= \tilde{u}_{1,4,\epsilon}^\infty(\hat{x}, d, k) \\
&= \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(z\cdot\theta + z'\cdot\theta')} \mathbf{A}_\epsilon\left(\frac{z+z'}{2}\right) \cdot \hat{x} \mathbf{A}_\epsilon\left(\frac{z-z'}{2}\right) \cdot d \frac{e^{ik|z'|} - e^{-k|z'|}}{|z'|} dz dz' \\
&= \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_0^\infty \int_{\mathbb{S}^2} e^{ik(z\cdot\theta + \rho\omega\cdot\theta')} \mathbf{A}_\epsilon\left(\frac{z+\rho\omega}{2}\right) \cdot \hat{x} \mathbf{A}_\epsilon\left(\frac{z-\rho\omega}{2}\right) \cdot d (e^{ik\rho} - e^{-k\rho}) \rho d\sigma(\omega) d\rho dz,
\end{aligned}$$

where $2\theta = d - \hat{x}$, $-2\theta' = \hat{x} + d$. By the Fatóu Lemma, the estimate

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{\frac{\sqrt{2}}{2}\mathbb{S}^2} \int_1^\infty k^{m_A+1} |u_{1,4,\epsilon}^\infty(\theta, \theta', k)|^2 dk d\sigma(\theta) < \infty \quad (4.15)$$

ensures the inequality

$$\mathbb{E} \int_{\frac{\sqrt{2}}{2}\mathbb{S}^2} \int_1^\infty k^{m_A+1} |u_{1,4}^\infty(\theta, \theta', k)|^2 dk d\sigma(\theta) < \infty.$$

As a consequence, we have

$$\int_1^\infty k^{m_A+1} |u_{1,4}^\infty(\theta, \theta', k)|^2 dk < \infty$$

which almost surely holds for almost every $\theta \in \frac{\sqrt{2}}{2}\mathbb{S}^2$ and any $\theta' \in \frac{\sqrt{2}}{2}\mathbb{S}^2$. Thus, for $j = 4$, (4.14) almost surely holds for all $(\hat{x}, d)$ in

$$\{(\hat{x}, d) \in \mathbb{S}^2 \times \mathbb{S}^2 : \langle \hat{x}, d \rangle = \pi/2, \hat{x} - d \notin S_0\},$$

where $S_0 \subset \sqrt{2}\mathbb{S}^2$ is a set of measure zero. Based on the above discussions, we only need to prove (4.15). Denote

$$f_\epsilon(z, \rho, k) = 1_{[0,\infty)}\rho \int_{\mathbb{S}^2} e^{ik\rho\omega\cdot\theta'} A_\epsilon\left(\frac{z+\rho\omega}{2}\right) \cdot \hat{x} A_\epsilon\left(\frac{z-\rho\omega}{2}\right) \cdot d\sigma(\omega)$$

and

$$Tf_\epsilon(z, k) = \int_{\mathbb{R}} (e^{ik\rho} + e^{-k\rho}) f_\epsilon(z, \rho, k) d\rho,$$

which gives

$$u_{1,4,\epsilon}^\infty(\theta, \theta', k) = \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{ikz\cdot\theta} Tf_\epsilon(z, k) dz. \quad (4.16)$$

Thanks to [2, Lemma 4.7], there holds the estimate

$$\int_{\frac{\sqrt{2}}{2}\mathbb{S}^2} k^2 \left| \int_{\mathbb{R}^3} e^{ikz\cdot\theta} Tf_\epsilon(z, k) dz \right|^2 d\sigma(\theta) \lesssim \int_{\mathbb{R}^3} |Tf_\epsilon(z, k)|^2 dz. \quad (4.17)$$

Combining (4.16)–(4.17), we obtain

$$\int_{\frac{\sqrt{2}}{2}\mathbb{S}^2} \int_1^\infty k^{m_A+1} |u_{1,4,\epsilon}^\infty(\theta, \theta', k)|^2 dk d\sigma(\theta) \lesssim \int_1^\infty k^{m_A-5} \int_{\mathbb{R}^3} |Tf_\epsilon(z, k)|^2 dz dk.$$

Hence, it suffices to show

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_1^\infty k^{m_A-5} \int_{\mathbb{R}^3} |Tf_\epsilon(z, k)|^2 dz dk < \infty. \quad (4.18)$$

Notice that

$$\begin{aligned} Tf_\epsilon(z, k) &= \int_0^\infty \int_{\mathbb{S}^2} (e^{ik\rho} + e^{-k\rho}) \rho e^{ik\rho\omega \cdot \theta'} A_\epsilon \left(\frac{z + \rho\omega}{2} \right) \cdot \hat{x} A_\epsilon \left(\frac{z - \rho\omega}{2} \right) \cdot d\sigma(\omega) d\rho \\ &= \int_{\mathbb{R}^3} (e^{ik|z'|} + e^{-k|z'|}) |z'|^{-1} e^{ikz' \cdot \theta'} A_\epsilon \left(\frac{z + z'}{2} \right) \cdot \hat{x} A_\epsilon \left(\frac{z - z'}{2} \right) \cdot d dz'. \end{aligned}$$

Applying the Isserli's theorem gives

$$\begin{aligned} &\mathbb{E} |Tf_\epsilon(z, k)|^2 \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (e^{ik|z'|} + e^{-k|z'|}) (e^{-ik|\tilde{z}'|} + e^{-k|\tilde{z}'|}) e^{ik(z' - \tilde{z}') \cdot \theta'} |z'|^{-1} |\tilde{z}'|^{-1} \mathbb{E}[X_1 X_2 X_3 X_4] dz' d\tilde{z}' \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (e^{ik|z'|} + e^{-k|z'|}) (e^{-ik|\tilde{z}'|} + e^{-k|\tilde{z}'|}) e^{ik(z' - \tilde{z}') \cdot \theta'} |z'|^{-1} |\tilde{z}'|^{-1} \mathbb{E}[X_1 X_2] \mathbb{E}[X_3 X_4] dz' d\tilde{z}' \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (e^{ik|z'|} + e^{-k|z'|}) (e^{-ik|\tilde{z}'|} + e^{-k|\tilde{z}'|}) e^{ik(z' - \tilde{z}') \cdot \theta'} |z'|^{-1} |\tilde{z}'|^{-1} \mathbb{E}[X_1 X_3] \mathbb{E}[X_2 X_4] dz' d\tilde{z}' \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (e^{ik|z'|} + e^{-k|z'|}) (e^{-ik|\tilde{z}'|} + e^{-k|\tilde{z}'|}) e^{ik(z' - \tilde{z}') \cdot \theta'} |z'|^{-1} |\tilde{z}'|^{-1} \mathbb{E}[X_1 X_4] \mathbb{E}[X_2 X_3] dz' d\tilde{z}' \\ &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |z'|^{-1} |\tilde{z}'|^{-1} (|\mathbb{E}[X_1 X_2] \mathbb{E}[X_3 X_4]| + |\mathbb{E}[X_1 X_3] \mathbb{E}[X_2 X_4]| + |\mathbb{E}[X_1 X_4] \mathbb{E}[X_2 X_3]|) dz' d\tilde{z}', \end{aligned}$$

where

$$X_1 = A_\epsilon \left(\frac{z + z'}{2} \right) \cdot \hat{x}, \quad X_2 = A_\epsilon \left(\frac{z - z'}{2} \right) \cdot d, \quad X_3 = A_\epsilon \left(\frac{z + \tilde{z}'}{2} \right) \cdot \hat{x}, \quad X_4 = A_\epsilon \left(\frac{z - \tilde{z}'}{2} \right) \cdot d.$$

We also have

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}[X_1 X_2] = \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[A_\epsilon \left(\frac{z+z'}{2} \right) \cdot \hat{x} A_\epsilon \left(\frac{z-z'}{2} \right) \cdot d \right] = \sum_{i,j=1}^3 K_{ij} \left(\frac{z+z'}{2}, \frac{z-z'}{2} \right) \hat{x}_i d_j,$$

where $K_{ij}(x, y) = (K_A(x, y))_{ij}$. Recalling Lemma 2.2, we know that for $m_A > 2$, $K_{ij}(x, y)$ is weakly singular, which yields that the above limit holds in point-wise sense. It can be verified that similar conclusions hold for all $\mathbb{E}[X_i X_j]$, $i, j = 1, 2, 3, 4$. Notice that the support of A is compact. When $2 < m_A < 4$, using Lemma 2.2, there exists $R_0 > 0$ such that

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{\mathbb{R}^3} \int_1^\infty k^{m_A-5} |Tf_\epsilon(z, k)|^2 dk dz &= \limsup_{\epsilon \rightarrow 0} \int_1^\infty k^{m_A-5} \int_{\mathbb{R}^3} \mathbb{E} |Tf_\epsilon(z, k)|^2 dz dk \\ &\lesssim \int_1^\infty k^{m_A-5} \int_{|z| < R_0} \int_{|z'| < R_0} \int_{|\tilde{z}'| < R_0} |z'|^{-1} |\tilde{z}'|^{-1} (|z'|^{-1} |\tilde{z}'|^{-1} + |z' - \tilde{z}'|^{-2} + |z' + \tilde{z}'|^{-2}) d\tilde{z}' dz' dz dk. \\ &< \infty. \end{aligned}$$

In summary, we have proven (4.15) for $j = 4$, which yields (4.13) for $j = 4$. The remaining cases of (4.13) for $j = 1, 2, 3$ can be treated in a similar way. $\square$

Proof of Lemma 4.5. Recall that the higher order terms are given by

$$u_2^\infty(\hat{x}, d, k) = -\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{-ik\hat{x} \cdot y} (A \cdot \nabla + V(y)) \sum_{j \geq 2} [(-R_0(k)(A \cdot \nabla + V(y)))^j u^{inc}](y) dy.$$

By applying the regularity result in Lemma 2.1, the resolvent estimates in Lemma 3.2, and the Sobolev inequality in Lemma 3.3 we have the estimate

$$\begin{aligned} |u_2^\infty(\hat{x}, d, k)| &\lesssim \frac{1}{k^2} \sum_{j \geq 2} \|e^{-ik\hat{x} \cdot y} \chi\|_{H^{-s}} \|((A \cdot \nabla + V(y)) R_0(k))^j (A \cdot \nabla + V(y)) e^{ik\hat{x} \cdot d}\|_{H^s} \\ &\lesssim \frac{1}{k^2} \sum_{j \geq 2} k^{-s} k^{-j(2+2s)} \|(A \cdot \nabla + V(y)) e^{ik\hat{x} \cdot d}\|_{H^s} \\ &\lesssim \frac{1}{k^2} \sum_{j \geq 2} k^{-s} k^{-j(2+2s)} \|e^{ik\hat{x} \cdot d}\|_{H^{-s+1}} \lesssim \sum_{j \geq 2} k^{-j(2+2s)-1-2s} \lesssim k^{-5-6s} \end{aligned}$$

for non-positive s such that $-1/2 < s < \min\{(m_A - 3)/2, (m_V - 3)/2\}$, which gives

$$k^{m_A+2} |u_2^\infty(\hat{x}_1, d_1, k)|^2 \lesssim k^{-(8+12s-m_A)}.$$

Hence, there holds

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_A+2} |u_2^\infty(\hat{x}_1, d_1, k)|^2 dk = 0$$

if s satisfies $8 + 12s - m_A > 0$ and $-1/2 < s < \min\{(m_A - 3)/2, (m_V - 3)/2\}$, i.e., $2 < m_A < 6m_V - 10$. $\square$

Proof of Theorem 4.1. Combining Lemma 4.4–4.5 and (4.12) gives (4.11), which implies (4.10). Then together with (4.10) and Lemma 4.2, the proof of Theorem 4.1 is completed. $\square$

4.2. Recovery of the zeroth-order perturbation

In this section, we determine the micro-correlation strength h of the potential V given the micro-correlation strength $\mathbb{H}$ from the backscattering far-field pattern. The following lemma shows that h can be determined from a single realization of the zeroth-order term of the far-field pattern together with $\mathbb{H}$.

Lemma 4.6. Let $1 + m_V < m_A$. Then for all $\tau > 0$ and $d \in \mathbb{S}^2$ there holds almost surely that

$$\begin{aligned} & \widehat{h}(\tau(\hat{x} - d)) \\ &= \lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} 64\pi^2 k^{m_V+4} u_0^\infty(-d, d, k + \tau) \overline{u_0^\infty(-d, d, k)} + \sum_{i,j=1}^3 k^{2+m_V-m_A} d_i d_j \mathbb{H}_{ij}(-2\tau\hat{x}) dk. \end{aligned}$$

Proof. By taking $\hat{x} = -d$ in (4.5) and taking the covariance, we have

$$\begin{aligned} & \mathbb{E}[u_0^\infty(-d, d, k + \tau) \overline{u_0^\infty(-d, d, k)}] \\ &= \frac{1}{64\pi^2 k^4} \left(-k^2 \mathbb{E} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (A(y) \cdot d A(z) \cdot d) e^{2i(k+\tau)y \cdot d - 2ikz \cdot d} dy dz \right. \\ & \quad \left. + \mathbb{E} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} V(y) V(z) e^{2i(k+\tau)y \cdot d - 2ikz \cdot d} dy dz \right). \end{aligned}$$

Then we have

$$\begin{aligned} & 64\pi^2 k^4 \mathbb{E}[u_0^\infty(-d, d, k + \tau) \overline{u_0^\infty(-d, d, k)}] \\ &= \mathbb{E}[\widehat{V}(-2(k + \tau)d) \widehat{V}(2kd)] - \sum_{i,j=1}^3 k^2 d_i d_j \mathbb{E}[\widehat{A}_j(-2(k + \tau)d) \widehat{A}_i(2kd)]. \end{aligned} \quad (4.19)$$

From (4.3) and (4.4) we obtain

$$\begin{aligned} & 64\pi^2 k^4 \mathbb{E}[u_0^\infty(-d, d, k + \tau) \overline{u_0^\infty(-d, d, k)}] + \sum_{i,j=1}^3 k^2 (d_i d_j \mathbb{H}_{ij}(-2\tau\hat{x}) k^{-m_A} + O(1/k^{1+m_A})) \\ &= h(-2\tau\hat{x}) k^{-m_V} + O(k^{-1-m_V}), \end{aligned}$$

which yields

$$\begin{aligned}
h(-2\tau\hat{x}) &= 64\pi^2 k^{4+m_V} \mathbb{E}[u_0^\infty(-d, d, k + \tau) \overline{u_0^\infty(-d, d, k)}] + O(k^{-1}) + O(k^{m_V+1-m_A}) \\
&\quad + \sum_{i,j=1}^3 k^{2+m_V-m_A} d_i d_j \mathbb{H}_{ij}(-2\tau\hat{x}).
\end{aligned} \tag{4.20}$$

From (4.20), similar ergodicity arguments as the proof of Lemma 4.2 complete the proof. $\square$

Following the arguments in Section 4.1, to further recover h from the far-field pattern, it suffices to prove

$$\lim_{k \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_V+4} |u_j^\infty(-d, d, k)|^2 dk = 0, \quad j = 1, 2.$$

Lemma 4.7. Assume $2 < m_V \leq 5$ and $m_A > m_V + 1$. Then

$$\lim_{k \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_V+4} |u_1^\infty(-d, d, k)|^2 dk = 0$$

almost surely holds for almost every $d \in \mathbb{S}^2$.

Lemma 4.8. Assume $m_V > \frac{12}{5}$, $m_V < 6m_A - 12$. Then there holds

$$\lim_{k \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_V+4} |u_2^\infty(-d, d, k)|^2 dk = 0$$

for all $d \in \mathbb{S}^2$.

Proof of Lemma 4.7. As in the proof of Lemma 4.4, we have

$$u_1^\infty(\hat{x}, d, k) = \frac{1}{8\pi} \sum_{j=1}^4 u_{1,j}^\infty(\hat{x}, d, k).$$

We first verify

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_V+4} |u_{1,4}^\infty(-d, d, k)|^2 dk = 0.$$

To end this, we start from the mollification

$$\begin{aligned}
& \tilde{u}_{1,4,\epsilon}^\infty(-d, d, k) \\
&= -\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ikz \cdot da} A_\epsilon \left(\frac{z+z'}{2} \right) \cdot dA_\epsilon \left(\frac{z-z'}{2} \right) \cdot d \frac{e^{ik|z'|} - e^{-k|z'|}}{|z'|} dz dz' \\
&= -\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_0^\infty \int_{\mathbb{S}^2} e^{ikz \cdot d} A_\epsilon \left(\frac{z+\rho\omega}{2} \right) \cdot dA_\epsilon \left(\frac{z-\rho\omega}{2} \right) \cdot d(e^{ik\rho} - e^{-k\rho}) \rho d\sigma(\omega) d\rho dz.
\end{aligned}$$

We want to prove

$$\mathbb{E} \int_{\mathbb{S}^2} \int_1^\infty k^{m_V+3} |u_{1,4}^\infty(-d, d, k)|^2 dk d\sigma(\omega) < \infty, \quad (4.21)$$

which almost surely gives

$$\int_1^\infty k^{m_V+3} |u_{1,4}^\infty(-d, d, k)|^2 dk < \infty$$

for almost every $d \in \mathbb{S}^2$. By the Fatóu Lemma, the estimate

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{\mathbb{S}^2} \int_1^\infty k^{m_V+3} |\tilde{u}_{1,4,\epsilon}^\infty(-d, d, k)|^2 dk d\sigma(d) < \infty \quad (4.22)$$

implies (4.21). To arrive at the estimate (4.22), we denote

$$f_\epsilon(z, \rho) = -1_{[0,\infty)} \rho \int_{\mathbb{S}^2} A_\epsilon \left(\frac{z+\rho\omega}{2} \right) \cdot dA_\epsilon \left(\frac{z-\rho\omega}{2} \right) \cdot d d\sigma(\omega)$$

and

$$Tf_\epsilon(z, k) = \int_{\mathbb{R}} (e^{ik\rho} + e^{-k\rho}) f_\epsilon(z, \rho) d\rho,$$

which gives

$$\tilde{u}_{1,4,\epsilon}^\infty(-d, d, k) = \frac{1}{8\pi k^2} \int_{\mathbb{R}^3} e^{ikz \cdot d} Tf_\epsilon(z, k) dz. \quad (4.23)$$

Applying [2, Lemma 4.7], there holds the estimate

$$\int_{\mathbb{S}^2} k^2 \left| \int_{\mathbb{R}^3} e^{ikz \cdot d} T f_\epsilon(z, k) dz \right|^2 d\sigma(d) \lesssim \int_{\mathbb{R}^3} |T f_\epsilon(z, k)|^2 dz. \quad (4.24)$$

Combining (4.23)–(4.24) yields

$$\int_{\mathbb{S}^2} \int_1^\infty k^{m_V+3} |\tilde{u}_{1,4,\epsilon}^\infty(-d, d, k)|^2 dk d\sigma(d) \lesssim \int_{\mathbb{R}^3} \int_1^\infty k^{m_V-3} |T f_\epsilon(z, k)|^2 dz dk. \quad (4.25)$$

Recall that

$$T f_\epsilon(z, k) = \int_{\mathbb{R}} (e^{ik\rho} + e^{-k\rho}) f_\epsilon(z, \rho) d\rho = (2\pi)^3 \mathcal{F}^{-1}(f_\epsilon(z, \cdot))(k) + \mathcal{L}(\tilde{f}_\epsilon(z, \cdot))(k) \quad (4.26)$$

with $\tilde{f}_\epsilon(z, \cdot) = f_\epsilon(z, \cdot)|_{[0,\infty)}$, where $\mathcal{L}(g)$ is the Laplace transform of g defined as follows

$$\mathcal{L}(g)(s) := \int_0^\infty g(x) e^{-sx} dx, \quad s > 0.$$

Then inserting the following identities

$$\mathcal{F}^{-1}(\partial_x g)(s) = -is \hat{g}(\xi), \quad \mathcal{L}(g)(s) = g(0) + \mathcal{L}(\partial_x g)(s)$$

into (4.25) and (4.26) gives

$$\begin{aligned} \int_{\mathbb{S}^2} \int_1^\infty k^{m_V+3} |\tilde{u}_{1,4,\epsilon}^\infty(-d, d, k)|^2 dk d\sigma(d) &\lesssim \int_{\mathbb{R}^3} \int_1^\infty k^{2N} |T f_\epsilon(z, k)|^2 dz dk \\ &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\rho^N f_\epsilon(z, \rho)|^2 d\rho dz, \end{aligned} \quad (4.27)$$

with N given by

$$N = \begin{cases} 0, & 2 < m_V \leq 3, \\ 1, & 3 < m_V \leq 5, \end{cases}$$

where we have also used the Plancherel identity and the boundedness of the Laplace transform in $L^2(\mathbb{R}^+)$. It is easy to verify that the support of f_ϵ is compact. Therefore, (4.27) implies that there exists $R_0 > 0$ such that

$$\begin{aligned}
& \limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{\frac{\sqrt{2}}{2} \mathbb{S}^2} \int_1^\infty k^{m_V+3} |\tilde{u}_{1,4,\epsilon}^\infty(-d, d, k)|^2 dk d\sigma(d) \\
& \lesssim \limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{|z| < R_0} \int_{|\rho| < R_0} |\partial_\rho^N f_\epsilon(z, \rho)|^2 d\rho dz.
\end{aligned} \tag{4.28}$$

Now to prove (4.22), one only needs to show that

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{|z| < R_0} \int_{|\rho| < R_0} |\partial_\rho^N f_\epsilon(z, \rho)|^2 d\rho dz < \infty. \tag{4.29}$$

To this end, we first consider the case $3 < m_V \leq 5$. Notice that there holds

$$\begin{aligned}
\partial_\rho f_\epsilon(z, \rho) &= -1_{[0,\infty)}(\rho) \int_{\mathbb{S}^2} A_\epsilon \left(\frac{z + \rho\omega}{2} \right) \cdot dA_\epsilon \left(\frac{z - \rho\omega}{2} \right) \cdot d\sigma(\omega) \\
&\quad - \rho 1_{[0,\infty)}(\rho) \int_{\mathbb{S}^2} \sum_{i,j=1}^3 \frac{d_i \omega_j}{2} \partial_j A_{\epsilon,i} \left(\frac{z + \rho\omega}{2} \right) A_\epsilon \left(\frac{z - \rho\omega}{2} \right) \cdot d\sigma(\omega) \\
&\quad + \rho 1_{[0,\infty)}(\rho) \int_{\mathbb{S}^2} \sum_{i,j=1}^3 \frac{d_i \omega_j}{2} \partial_j A_{\epsilon,i} \left(\frac{z - \rho\omega}{2} \right) A_\epsilon \left(\frac{z + \rho\omega}{2} \right) \cdot d\sigma(\omega) \\
&:= f_{1,\epsilon}(z, \rho) + f_{2,\epsilon}(z, \rho) + f_{3,\epsilon}(z, \rho).
\end{aligned}$$

Taking covariance gives

$$\mathbb{E} |\partial_\rho f_\epsilon(z, \rho)|^2 = \sum_{i,j=1}^3 \mathbb{E} [f_{i,\epsilon}(z, \rho) \overline{f_{j,\epsilon}(z, \rho)}].$$

We first consider $\mathbb{E} [f_{3,\epsilon}(z, \rho) \overline{f_{3,\epsilon}(z, \rho)}]$. In fact, by the Isserli's theorem there holds

$$\begin{aligned}
\mathbb{E} [f_{3,\epsilon}(z, \rho) \overline{f_{3,\epsilon}(z, \rho)}] &= \rho^2 1_{[0,\infty)} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} [Y_1 Y_2 Y_3 Y_4] d\sigma(\omega) d\sigma(\tilde{\omega}) \\
&= \rho^2 1_{[0,\infty)}(\rho) \left\{ \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} [Y_1 Y_2] \mathbb{E} [Y_3 Y_4] d\sigma(\omega) d\sigma(\tilde{\omega}) \right. \\
&\quad + \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} [Y_1 Y_3] \mathbb{E} [Y_2 Y_4] d\sigma(\omega) d\sigma(\tilde{\omega}) \\
&\quad \left. + \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \mathbb{E} [Y_1 Y_4] \mathbb{E} [Y_2 Y_3] d\sigma(\omega) d\sigma(\tilde{\omega}) \right\},
\end{aligned} \tag{4.30}$$

where

$$Y_1 := \sum_{i,j=1}^3 \frac{d_i \omega_j}{2} \partial_j A_{\epsilon,i} \left(\frac{z - \rho \omega}{2} \right), \quad Y_2 := A_{\epsilon} \left(\frac{z + \rho \omega}{2} \right) \cdot d,$$

$$Y_3 := \sum_{i,j=1}^3 \frac{d_i \tilde{\omega}_j}{2} \partial_j A_{\epsilon,i} \left(\frac{z - \rho \tilde{\omega}}{2} \right), \quad Y_4 := A_{\epsilon} \left(\frac{z + \rho \tilde{\omega}}{2} \right) \cdot d.$$

We also have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbb{E}[Y_1 Y_3] &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\sum_{i,j=1}^3 \frac{d_i \omega_j}{2} \partial_j A_{\epsilon,i} \left(\frac{z - \rho \omega}{2} \right) \sum_{l,s=1}^3 \frac{d_s \tilde{\omega}_l}{2} \partial_l A_{\epsilon,s} \left(\frac{z - \rho \tilde{\omega}}{2} \right) \right] \\ &= \sum_{i,j,l,s=1}^4 \frac{d_i \omega_j d_s \tilde{\omega}_l}{4} \partial_{1,j} \partial_{2,l} K_{i,s} \left(\frac{z - \rho \omega}{2}, \frac{z - \rho \tilde{\omega}}{2} \right), \end{aligned}$$

where $\partial_{1,j} \partial_{2,l} K_{i,s}(x, y) := \partial_{x_j} \partial_{y_l} K_{i,s}(x, y)$. Recalling Lemma 2.2, we know that for $m_A > 2$, $\partial_{1,j} \partial_{2,l} K_{i,s}$ is weakly singular. Thus, the above limit holds in point-wise sense. Similar conclusions hold for the other $\mathbb{E}[Y_i Y_j]$ ($i, j = 1, 2, 3, 4$). Furthermore, as $m_V \in (3, 5]$ and $m_A > m_V + 1$ implies that $m_A > 4$, from (4.30) we have

$$\begin{aligned} &\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{|z| < R_0} \int_{|\rho| < R_0} [f_{3,\epsilon}(z, \rho) f_{3,\epsilon}(z, \rho)] d\rho dz \\ &\lesssim \int_{|z| < R_0} \int_0^{R_0} \rho^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} |\rho \omega|^{m_A-4} |\rho \tilde{\omega}|^{m_A-4} + |\rho(\omega - \tilde{\omega})|^{m_A-5} \\ &\quad + |\rho(\omega + \tilde{\omega})|^{2(m_A-4)} d\sigma(\tilde{\omega}) d\sigma(\omega) d\rho dz < \infty, \end{aligned}$$

which completes the proof for $m_V \in (3, 5]$. Similarly we can verify

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E} \int_{|z| < R_0} \int_{|\rho| < R_0} [f_{i,\epsilon}(z, \rho) f_{j,\epsilon}(z, \rho)] d\rho dz < \infty, \quad i, j = 1, 2, 3. \quad (4.31)$$

Moreover, we can obtain (4.31) for $m_V \in (2, 3)$ in a similar way. Hence, we have (4.29) which yields (4.22). Using similar arguments as above we have the following limits for the remaining cases

$$\lim_{K \rightarrow \infty} \frac{1}{K} \int_K^{2K} k^{m_V+4} |u_{1,j}^\infty(-d, d, k)|^2 dk = 0, \quad j = 1, 2, 3,$$

which complete the proof. $\square$

Proof of Lemma 4.8. Recalling the proof of Lemma 4.5, we have known that

$$k^{m_V+4}|u_2^\infty(-d, d, k)|^2 \lesssim k^{-(6+12s-m_V)}$$

for non-positive s with $-1/2 < s < \min\{(m_A - 3)/2, (m_V - 3)/2\}$. Thus, we have the following lemma. $\square$

Proof of Theorem 4.2. Combining Lemma 4.6–4.8 and (4.12) completes the proof. $\square$

To end this section, we briefly discuss the possibility of extending the arguments in this paper to the two-dimensional case. Since the Green function in the two-dimensional case is $\frac{1}{4}H_0^{(1)}(k|x-y|)$, we can not directly use the analysis in Lemma 4.5 and Lemma 4.7. A natural idea is to apply the asymptotic expansion of the Hankel function as follows [13, (10.17.13)]:

$$H_0^{(1)}(k|x-y|) = \sqrt{\frac{1}{z}} e^{i(k|x-y|-\frac{\pi}{4})} \sum_{j=0}^N a_j(k|x-y|)^{-j} + \mathcal{O}((k|x-y|)^{-N-1})$$

as $k|x-y| \rightarrow \infty$. However, recalling the first order term of the Born series of the far-field pattern $u^\infty(\hat{x}, d, k)$, which consists of the term

$$\frac{1}{8\pi k^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ik(d \cdot s - \hat{x} \cdot y)} G(y, s) V(y) V(s) \, ds \, dy$$

and other similar terms. Hence the above asymptotic relation can not be applied due to the singularity at $y = s$. So it seems not straightforward to extend arguments in this paper to the two-dimensional case. We can still expect to use other measurements to recover the random potential in the two-dimensional case. In fact, for the second order randomly perturbed elliptic operator, the measurements in a domain far away from the random potential can be used to uniquely recover the random potential [9]. It is an interesting problem to extend the arguments [9] to the model in this paper.

5. Conclusion

In this paper, we have investigated the direct and inverse scattering problems for the biharmonic operator, incorporating a stochastic first-order perturbation. Our study encompasses the meromorphic continuation of the resolvent for the biharmonic operator, particularly with rough potentials. We have successfully derived a logarithmic analytic domain and corresponding resolvent estimates, which facilitate the derivation of the Born series expansion for the far-field pattern. Based upon this expansion, we further investigate the uniqueness of inverse scattering problem, i.e., to recover the principal symbol of the covariance. Our findings reveal that the micro-correlation strengths of two random potentials can be simultaneously determined through a single instance of the far-field pattern, which is averaged over a frequency band almost surely. Additionally, we have demonstrated that the uniqueness of the solution can be achieved by measuring the far-field pattern within a finite frequency interval. The discussion in this paper is applicable to other various wave equations including Helmholtz equation, Navier equation and Maxwell equation. However, the decay of resolvent of the second order operator with respect

to the frequency is slower than biharmonic operator, which brings more difficulties to guarantee the vanishing the remainder terms of the far-field pattern as the frequency goes to infinity. Moreover, another interesting future direction is to study the stability of recovering micro-correlation strengths from near-field data.

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Data availability

No data was used for the research described in the article.

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