

RESEARCH ARTICLE

Resonance Analysis of One-Dimensional Acoustic Media: A Propagation Matrix Approach

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ABSTRACT

The propagation matrix is a natural and powerful tool for analyzing one-dimensional wave propagation. While it has been widely applied to study wave phenomena in specific one-dimensional media, a systematic framework based on this method for characterizing scattering resonances and deriving their asymptotics in general settings has been lacking. This work fills this gap by developing such a framework for general one-dimensional acoustic media. Specifically, by combining the propagation matrix with Möbius transformations, we characterize resonant frequencies as the zeros of an explicit trigonometric polynomial. Leveraging Nevanlinna's value distribution theory, we establish the distribution properties of the resonances and demonstrate that their imaginary parts are uniformly bounded, which contrasts with the three-dimensional case. In two classes of high-contrast regimes, we derive the asymptotics of both subwavelength and non-subwavelength resonances with respect to the contrast parameter. Furthermore, by applying the Newton polygon method, we recover the discrete capacitance matrix approximation for subwavelength Minnaert resonances in both Hermitian and non-Hermitian cases, thereby establishing its connection to the propagation matrix framework. In this way, we provide a new and self-contained framework, distinct from variational methods (Feppon et al., *SIAM Journal on Applied Mathematics*, 83, no. 2 (2023): 625–665), for analyzing the asymptotics of one-dimensional scattering resonances and facilitating the analysis of high-frequency resonances.

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1 | Introduction

A fundamental principle in wave physics asserts that, for an object or structure to interact strongly with a wave, such as inducing significant scattering or refraction, its size must be comparable to the wavelength. This principle underpins phenomena like the Abbe diffraction limit, which defines the resolution limit of optical systems, and guides the design of radio antennas. A major scientific challenge is, therefore, to enable wave manipulation at scales far smaller than the wavelength. This has sparked significant interest in the phenomenon of subwavelength resonance, where resonators

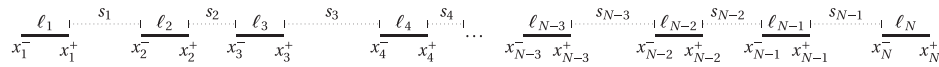


FIGURE 1 | A chain of N resonators, with lengths $(\ell_j)_{1 \leq j \leq N}$ and spacings $(s_j)_{1 \leq j \leq N-1}$.

exhibit strong interactions with incident waves whose wavelengths are orders of magnitude larger than the resonator itself.

A key mechanism for achieving subwavelength resonance is the use of high-contrast media, which consist of structures formed by embedding bounded inclusions with properties that differ significantly from those of the surrounding medium. The stark contrast between the inclusions and the background is a critical prerequisite for the emergence of subwavelength resonance [1, 2]. A classic example of this phenomenon is the Minnaert resonance, observed in air bubbles immersed in water [3]. Similar subwavelength resonances also appear in other high-contrast systems, including dielectric particles [4], plasmonic particles [5], and Helmholtz resonators [6]. The excitation of these resonances has enabled a wide range of innovative wave-based applications, such as superfocusing [7, 8], cloaking [9, 10], and wave guiding [11, 12].

In terms of mathematical analysis, in the high-contrast regime, it has been proven for two- and three-dimensional cases, using layer potential techniques, that resonances exist in the subwavelength regime, giving rise to the so-called Minnaert resonance [1, 13]. Furthermore, the leading-order terms in their asymptotics with respect to the contrast have been shown to be characterized by the eigenvalues of the capacitance matrix [14]. However, the boundary integral equation approach is not applicable to one-dimensional systems. Recently, Feppon et al. [15] provided a rigorous analysis of subwavelength resonances in one-dimensional acoustic wave scattering problems, utilizing the variational framework developed in [16]. On the other hand, the propagation matrix approach, which is particularly well-suited for one-dimensional Helmholtz equations (ODEs), has been widely employed to analyze resonant wave propagation in topological or disordered media. In [17–19], the propagation matrix method is used to study topological properties of periodic structures and periodic structures with defects. In [20, 21], the propagation matrix is combined with techniques from dynamical systems theory to characterize localized modes in certain disordered systems. Nevertheless, a general propagation matrix framework for scattering resonances and their asymptotics in one-dimensional acoustic media consisting of resonators with arbitrary lengths, spacings, and material parameters has yet to be developed.

In this work, we investigate the scattering resonances of acoustic waves in one-dimensional media, where the resonators and the background form a finite chain of segments with arbitrary lengths, interdistances, and material properties (see Figure 1). We develop a propagation matrix framework for analyzing general scattering resonances and their asymptotics. In particular, our framework recovers the capacitance matrix approximation in the subwavelength regime [15], thereby establishing the connection between the two approaches. Compared to the variational method in [15], our propagation matrix approach enables a more direct analysis of resonance asymptotics and resonant modes; see [22, 23] for its recent applications to the discrete approximation of and exceptional points in non-subwavelength resonances.

1.1 | Main Results

We consider the one-dimensional acoustic resonance problem (2.4) with the material parameters given by (2.2) and (2.3). In this work, we fix the wave-speed ratio r and examine the dependence of the resonances on the complex density ratio δ . Our main contributions are summarized as follows.

First, using Möbius transformations, we derive an equivalent analytical condition (2.17) for the resonant frequencies in terms of the propagation matrix. In Theorem 2.2, we show that the system exhibits identical resonant frequencies for the density ratios δ and r^2/δ , and that when $\delta \in \mathbb{R}$, all resonant frequencies are symmetric with respect to the imaginary axis. In particular, when $\delta > 0$, all resonant frequencies lie in the lower half of the complex plane.

Second, using the analytical formulation (2.17), we explicitly compute the analytic function $f(k; \sigma)$, with $\sigma = \delta/r$, whose zeros characterize all the resonant frequencies (Theorem 3.1). Then, we establish the general distribution pattern of these zeros and characterize the resonance-free region in Theorem 3.3, by applying Nevanlinna value distribution theory. Moreover, we prove that as $\delta \rightarrow 0$, the zeros of $f(\cdot; \sigma)$, and hence the resonant frequencies, converge to the set $E = \bigcup_{j=1}^{2N-1} (\pi\mathbb{Z}/t_j)$. The number of zeros near each $k_0 \in E$ is determined by the multiplicity $n(k_0)$ of the corresponding zero of a limiting analytic function $f(\cdot; 0)$ (see Theorem 3.5). For the case of simple zeros ($n(k_0) = 1$), we employ the implicit function theorem to derive the first-order asymptotics of the corresponding resonances as $\delta \rightarrow 0$ and $\delta \rightarrow \infty$ in Theorem 3.6.

Third, we further investigate the asymptotic behavior of subwavelength resonances as $\delta \rightarrow 0$ and $\delta \rightarrow \infty$ in Theorems 4.4 and 5.1 for Hermitian and non-Hermitian systems, respectively. Unlike the expansion in the previous section, where the implicit function theorem was applicable, the characteristic function $f(k; \sigma)$ in (3.3) possesses a high-order root at zero, necessitating the use of the Newton polygon method (see Appendix A) from multivariate complex analysis. This enables us to recover the capacitance matrix theory [15] from a novel complex analytic perspective. Moreover, for subwavelength resonant modes, we find that as $\delta \rightarrow 0$, the eigenmodes are approximately constant within the resonators and nearly linear within the spacing layers, with amplitudes governed by the capacitance matrix eigenvectors. Conversely, as $\delta \rightarrow \infty$, the eigenmodes are approximately linear within the resonators and nearly constant within the spacings. Another interesting finding is that subwavelength resonances persist in one-dimensional systems as $\delta \rightarrow \infty$, yet are entirely absent in three-dimensional structures (Section 4.3).

1.2 | Outlines

The paper is organized as follows. In Section 2, we characterize the resonances in one-dimensional acoustic media using the propagation matrix method. In Section 3, we establish the general distribution properties of the resonances and derive the asymptotics for two high-contrast regimes. Then, in Section 4, we present how the capacitance matrix approximation for subwavelength resonances can be recovered within the propagation matrix framework. Finally, in Section 5, we generalize the results from Section 4 to nonreciprocal systems.

2 | Preliminaries

This section introduces the scattering resonance problem for acoustic waves in one-dimensional inhomogeneous media. We employ the propagation matrix method to characterize the resonant frequencies. The main result, Theorem 2.2, establishes fundamental properties of scattering resonances and unveils a duality between two distinct contrast regimes.

2.1 | Model Setting

We consider a one-dimensional chain of N disjoint, identical resonators $D_j := (x_j^-, x_j^+)$, where $(x_j^\pm)_{1 \leq j \leq N} \subset \mathbb{R}$ are the $2N$ boundary points satisfying $x_j^- < x_j^+ < x_{j+1}^-$ for all $1 \leq j \leq N-1$. The length of each resonator is denoted by $\ell_j = x_j^+ - x_j^-$, and the spacing between the j th and $(j+1)$ th inclusions is given by $s_j = x_{j+1}^- - x_j^+$. The configuration of the system is illustrated in Figure 1.

We denote the collection of subwavelength resonators by the set

$$D := \bigcup_{j=1}^N (x_j^-, x_j^+).$$

In this work, we study the one-dimensional Helmholtz equation for the acoustic wave propagation in a heterogeneous medium associated with D :

$$\frac{\omega^2}{\kappa(x)} u(x) + \frac{d}{dx} \left(\frac{1}{\rho(x)} \frac{d}{dx} u(x) \right) = 0, \quad x \in \mathbb{R}, \quad (2.1)$$

where the bulk modulus $\kappa(x)$ and the density $\rho(x)$ of the medium are assumed to be piecewise constant inside and outside the resonators:

$$\kappa(x) = \begin{cases} \kappa_b, & x \in D, \\ \kappa, & x \in \mathbb{R} \setminus D, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_b, & x \in D, \\ \rho, & x \in \mathbb{R} \setminus D. \end{cases} \quad (2.2)$$

The wave speeds inside the set D of resonators and in the background $\mathbb{R} \setminus D$ are denoted by v_b and v , respectively, with the corresponding wavenumbers k_b and k . The contrasts between the densities and wave speeds of the resonators and the

background medium are denoted by δ and r , respectively. Specifically, let

$$v_b := \sqrt{\frac{\kappa_b}{\rho_b}}, \quad v := \sqrt{\frac{\kappa}{\rho}}, \quad k_b := \frac{\omega}{v_b}, \quad k := \frac{\omega}{v}, \quad \delta := \frac{\rho_b}{\rho}, \quad r := \frac{v}{v_b}. \quad (2.3)$$

Due to their physical significance, the parameters defined in (2.3) are typically treated as positive real numbers. However, as established in Theorem 3.1, the resonant frequencies of the system are uniquely determined by the parameters δ and r , which can, in general, be complex. In this work, we fix r as a positive constant, allow $\delta \in \mathbb{C}$, and study the dependence of the resonance ω on the complex parameter δ .

Using the parameters defined in (2.3), the Helmholtz equation (2.1) can be reformulated as follows:

$$\begin{cases} \frac{d^2}{dx^2} u(x) + k^2 u(x) = 0, & x \in \mathbb{R} \setminus D, \\ \frac{d^2}{dx^2} u(x) + k_b^2 u(x) = 0, & x \in D, \\ u|_+(x_j^\pm) = u|_-(x_j^\pm), & 1 \leq j \leq N, \\ \frac{du}{dx}|_\pm(x_j^\mp) = \delta \frac{du}{dx}|_\mp(x_j^\mp), & 1 \leq j \leq N, \\ \left(\frac{d}{d|x|} - ik\right)u = 0 & \text{for } x \in (-\infty, x_1^-) \cup (x_N^+, +\infty), \end{cases} \quad (2.4)$$

where for a one-dimensional function w , we denote its left and right limits, if they exist, by $w|_\pm(x) := \lim_{s \rightarrow 0^+} w(x \pm s)$. We say that $\omega \in \mathbb{C}$ is a *resonant frequency (resonance)* if the scattering problem (2.4) admits nontrivial solutions u , which are called resonant modes.

2.2 | Propagation Matrix and Möbius Transformation

For the scattering problem (2.4), the zero frequency, $\omega^* = 0$, is always a resonant frequency, as any constant function satisfies the equation. In this work, however, we focus on the nontrivial resonant frequencies ($\omega \neq 0$). To characterize these nontrivial resonances, we introduce the propagation matrix method in this section.

Let us begin with the following second-order ODE for $0 \neq k \in \mathbb{C}$:

$$\frac{d^2}{dx^2} u(x) + k^2 u(x) = 0, \quad x \in (0, a), \quad (2.5)$$

where the solution can be expressed as $u(x) = Ae^{ikx} + Be^{-ikx}$, with $A, B \in \mathbb{C}$ uniquely given by

$$A = \frac{u(0) + \frac{1}{ik}u'(0)}{2}, \quad B = \frac{u(0) - \frac{1}{ik}u'(0)}{2}.$$

It follows that

$$\begin{aligned} u(a) &= Ae^{ika} + Be^{-ika} = \cos(ka)u(0) + \frac{1}{k}\sin(ka)u'(0), \\ u'(a) &= ikAe^{ika} - ikBe^{-ika} = -k\sin(ka)u(0) + \cos(ka)u'(0). \end{aligned} \quad (2.6)$$

Define matrices, for $z, k \in \mathbb{C}$ and $a \in \mathbb{R}$,

$$T(k, a) := \begin{pmatrix} \cos(ka) & \sin(ka) \\ -\sin(ka) & \cos(ka) \end{pmatrix}, \quad M(z) := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}. \quad (2.7)$$

The propagation matrix $P(k, a)$ for (2.5) can be defined by (2.6) as follows:

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = P(k, a) \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}, \quad P(k, a) := M(k)T(k, a)M\left(\frac{1}{k}\right). \quad (2.8)$$

We now turn to the scattering problem (2.4), and the above formula (2.8) implies

$$\begin{pmatrix} u|_-(x_{j+1}^-) \\ u'|_-(x_{j+1}^-) \end{pmatrix} = P(k, s_j) \begin{pmatrix} u|_+(x_j^+) \\ u'|_+(x_j^+) \end{pmatrix}, \quad 1 \leq j \leq N-1, \quad (2.9)$$

and

$$\begin{pmatrix} u|_-(x_j^+) \\ u'|_-(x_j^+) \end{pmatrix} = P(rk, \ell_j) \begin{pmatrix} u|_+(x_j^-) \\ u'|_+(x_j^-) \end{pmatrix}, \quad 1 \leq j \leq N. \quad (2.10)$$

Note from the transmission boundary condition in (2.4) that for $1 \leq j \leq N$,

$$\begin{pmatrix} u|_+(x_j^+) \\ u'|_+(x_j^+) \end{pmatrix} = M\left(\frac{1}{\delta}\right) \begin{pmatrix} u|_-(x_j^+) \\ u'|_-(x_j^+) \end{pmatrix}, \quad \begin{pmatrix} u|_+(x_j^-) \\ u'|_+(x_j^-) \end{pmatrix} = M(\delta) \begin{pmatrix} u|_-(x_j^-) \\ u'|_-(x_j^-) \end{pmatrix}. \quad (2.11)$$

Combining (2.10) and (2.11) gives

$$\begin{pmatrix} u|_+(x_j^+) \\ u'|_+(x_j^+) \end{pmatrix} = M\left(\frac{rk}{\delta}\right)T(rk, \ell_j)M\left(\frac{\delta}{rk}\right) \begin{pmatrix} u|_-(x_j^-) \\ u'|_-(x_j^-) \end{pmatrix}, \quad 1 \leq j \leq N, \quad (2.12)$$

thanks to $M(z_1)M(z_2) = M(z_1z_2)$. By the radiation condition in (2.4), there holds

$$u(x) = \begin{cases} c_1 e^{-ik(x-x_1^-)}, & x < x_1^-, \\ c_2 e^{ik(x-x_N^+)}, & x > x_N^+, \end{cases}$$

with $c_1, c_2 \in \mathbb{C}$. We assume both c_1 and c_2 are nonzero; otherwise, $u \equiv 0$. Consequently, we have, using $M(z)$ in (2.7),

$$\begin{pmatrix} u_-(x_1^-) \\ u'|_-(x_1^-) \end{pmatrix} = c_1 M(k) \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \begin{pmatrix} u_+(x_N^+) \\ u'|_+(x_N^+) \end{pmatrix} = c_2 M(k) \begin{pmatrix} 1 \\ i \end{pmatrix}. \quad (2.13)$$

Thus, for a nontrivial resonance $\omega = kv \neq 0$, Equations (2.9), (2.12), and (2.13) imply

$$c \begin{pmatrix} 1 \\ i \end{pmatrix} = M\left(\frac{r}{\delta}\right)T(rk, l_N)M\left(\frac{\delta}{r}\right)T(k, s_{N-1}) \cdots M\left(\frac{r}{\delta}\right)T(kr, l_1)M\left(\frac{\delta}{r}\right) \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad (2.14)$$

where $c = c_2/c_1 \neq 0$. We summarize the above discussion as follows.

Lemma 2.1. $\omega \in \mathbb{C} \setminus \{0\}$ is a resonance for (2.4) if and only if the corresponding wavenumber $k = \omega/v$ satisfies (2.14) for some $c \neq 0$. In this case, c is uniquely determined by k .

We introduce a vector

$$\mathbf{t} := (r\ell_1, s_1, r\ell_2, s_2, \dots, r\ell_{N-1}, s_{N-1}, r\ell_N)^\top \in \mathbb{R}_{>0}^{2N-1}, \quad (2.15)$$

and denote

$$\sigma = \frac{\delta}{r}.$$

Using $T(rk, a) = T(k, ra)$ by definition (2.7), Equation (2.14) can be written as

$$c \begin{pmatrix} 1 \\ i \end{pmatrix} = M\left(\frac{1}{\sigma}\right) T(k, t_{2N-1}) M(\sigma) T(k, t_{2N-2}) \cdots M\left(\frac{1}{\sigma}\right) T(k, t_1) M(\sigma) \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad (2.16)$$

where t_j is the j th component of $\mathbf{t}$. Letting $e_{\pm} = (1, \pm i)^{\top}$, we have, by (2.7),

$$T(k, a)e_{\pm} = e^{\pm ika} e_{\pm}, \quad M(\sigma)e_{\pm} = \frac{1+\sigma}{2} e_{\pm} + \frac{1-\sigma}{2} e_{\mp}.$$

This enables us to further rewrite (2.16) as: for $c \neq 0$,

$$c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R\left(\frac{1}{\sigma}\right) L(t_{2N-1}k) R(\sigma) L(t_{2N-2}k) \cdots R\left(\frac{1}{\sigma}\right) L(t_1k) R(\sigma) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (2.17)$$

where for $z \in \mathbb{C}$,

$$R(z) := \begin{pmatrix} \frac{1+z}{2} & \frac{1-z}{2} \\ \frac{1-z}{2} & \frac{1+z}{2} \end{pmatrix}, \quad L(z) := \begin{pmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{pmatrix}. \quad (2.18)$$

The characterization in (2.17) is fundamental to all the main results of this paper. As a first application, we next establish some general properties of the resonant frequencies, with a more detailed treatment reserved for Sections 3 and 4.

We define the Möbius transformation (see [24, Chapter 3]) associated with a complex matrix $A = (a_{ij})_{2 \times 2} \in \mathbb{C}^{2 \times 2}$ by

$$f_A(z) := \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}, \quad z \in \mathbb{C} \cup \{\infty\}, \quad (2.19)$$

and for $\Phi = (\phi_1, \phi_2)^{\top} \in \mathbb{C}^2$, we define $\mathcal{M}(\Phi) := \phi_1/\phi_2 \in \mathbb{C} \cup \{\infty\}$. It is straightforward to verify that

$$\mathcal{M}(A\Phi) = f_A \circ \mathcal{M}(\Phi), \quad f_{AB} = f_A \circ f_B. \quad (2.20)$$

We then introduce two classes of rational functions by (2.19), based on matrices $R(\sigma)$ and $L(t_jk)$,

$$f_{\sigma}(z) := f_{R(\sigma)}(z) = \frac{(1+\sigma)z + (1-\sigma)}{(1-\sigma)z + (1+\sigma)}, \quad g_j(z) := f_{L(t_jk)}(z) = e^{2ikt_j} z. \quad (2.21)$$

Applying $\mathcal{M}$ on both sides of (2.17) and using (2.20)–(2.21), we obtain

$$f_{\sigma^{-1}} \circ g_{2N-1} \circ f_{\sigma} \circ \cdots \circ g_1 \circ f_{\sigma} \circ \mathcal{M}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathcal{M}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right),$$

that is,

$$f_{\sigma^{-1}} \circ g_{2N-1} \circ f_{\sigma} \circ \cdots \circ g_1 \circ f_{\sigma}(0) = \infty, \quad (2.22)$$

which is equivalent to (2.17). We now conclude this section with the following theorem.

Theorem 2.2. *For resonant frequencies ω of the problem (2.4), we have*

1. *If $\delta \in \mathbb{R}$, all resonances are symmetric with respect to the imaginary axis.*
2. *If $\delta > 0$, all nontrivial resonant frequencies have negative imaginary parts.*

3. Let $0 \neq \delta_0 \in \mathbb{C}$, then (2.4) with $\delta = \delta_0$ and $\delta = \frac{r^2}{\delta_0}$ has the same resonances $\omega = kv$. Moreover, let $u(x)$ and $v(x)$ be the resonant modes associated with a resonance ω for $\delta = \delta_0$ and $\delta = \frac{r^2}{\delta_0}$, respectively, satisfying $u(x) = e^{-i\frac{\omega}{v}x}$, $v(x) = te^{-i\frac{\omega}{v}x}$, $x < x_1^-$ for some $0 \neq t \in \mathbb{C}$. Then,

$$\begin{aligned} u'(x) &= -i\frac{\delta}{tr}k_b v(x), & v'(x) &= -i\frac{tr}{\delta}k_b u(x), & x &\in D \\ u'(x) &= -it^{-1}kv(x), & v'(x) &= -itku(x), & x &\in \mathbb{R} \setminus D, \end{aligned} \quad (2.23)$$

where, at each endpoint $x_j^\pm$, $u'(x)$ is interpreted as either $\frac{du}{dx}\big|_-$ or $\frac{du}{dx}\big|_+$.

Remark. Statement (3) asserts that, in one-dimensional space, a system with high-density resonators embedded in a low-density background shares exactly the same set of resonant frequencies as the system with the density contrast reversed. However, this property is unique to one dimension and does not generally hold in higher-dimensional spaces. Specifically, we will demonstrate its failure in the three-dimensional case in Section 4.3.

Proof of Theorem 2.2. For (1), we only need to take the conjugate of both sides of (2.22) which shows $(\delta, -\bar{k})$ also satisfies (2.22).

For (2), it relies on the fact that $|f_\sigma(z)| < 1$ for $|z| < 1$, which can be proved by applying the following two properties: $|z| < 1$ if and only if $\Re(\frac{z-1}{z+1}) < 0$; and $\frac{f_\sigma(z)-1}{f_\sigma(z)+1} = \sigma \frac{z-1}{z+1}$. Moreover, if $\Im m(k) \geq 0$, noting $r > 0$, then $|g_j(z)| \leq |z|$ for any $z \in \mathbb{C}$. Therefore, the modulus of the left-hand side of (2.22) must always remain less than 1 and can never approach ∞ . It follows that $\Im m(k) < 0$.

For the first part of (3), the proof relies on the key observation $f_\sigma(-z) = -f_{\sigma^{-1}}(z)$. Then, for (δ, k) satisfying (2.22), we have

$$\begin{aligned} & f_\sigma \circ g_{2N-1} \circ f_{\sigma^{-1}} \circ \cdots \circ g_1 \circ f_{\sigma^{-1}}(0) \\ &= f_\sigma \circ g_{2N-1} \circ f_{\sigma^{-1}} \circ \cdots \circ g_1 \circ -f_\sigma(0) \\ &= f_\sigma \circ g_{2N-1} \circ f_{\sigma^{-1}} \circ \cdots \circ (-g_1) \circ f_\sigma(0) \\ &= \cdots \\ &= -f_{\sigma^{-1}} \circ g_{2N-1} \circ f_\sigma \circ \cdots \circ g_1 \circ f_\sigma(0) = \infty. \end{aligned}$$

Thus, $(\frac{r^2}{\delta}, k)$ also satisfies (2.22), completing the proof.

For the second part of (3), we first show the following claim:

Claim: Assume $0 \neq k \in \mathbb{C}$, and the functions u and v satisfy the Helmholtz equation $w'' + k^2 w = 0$ on the interval $[0, a]$. If the boundary conditions $u'(0) = iktv(0)$ and $v'(0) = ikt^{-1}u(0)$ hold at the endpoint 0 for some $t \in \mathbb{C}$, then for all $x \in [0, a]$, we have $u'(x) = iktv(x)$ and $v'(x) = ikt^{-1}u(x)$.

The claim is derived by using the propagation matrix in (2.8). Specifically, we have

$$\begin{aligned} u(x) &= \cos(kx)u(0) + \frac{1}{k} \sin(kx)u'(0) = \cos(kx)\frac{t}{ik}v'(0) + \frac{1}{k} \sin(kx)ikt v(0) \\ &= \frac{t}{ik} (\cos(kx)v'(0) - k \sin(kx)v(0)) = \frac{t}{ik} v'(x). \end{aligned}$$

Similarly, using the same approach, we can derive that $u'(x) = iktv(x)$. According to the above claim, to prove the relations (2.23) between the eigenmodes, it suffices to show that (2.23) holds at every endpoint $x_j^\pm$. By the assumption on $u(x)$ and

$v(x)$, (2.23) is satisfied on $(-\infty, x_1^-)$. Then, using the jump relation, we have

$$\begin{aligned} \frac{du}{dx} \Big|_+ (x_1^-) &= \delta \frac{du}{dx} \Big|_- (x_1^-) = -i \frac{\delta}{tr} k_b v(x), \\ \frac{dv}{dx} \Big|_+ (x_1^-) &= \frac{r^2}{\delta} \frac{dv}{dx} \Big|_- (x_1^-) = -i \frac{tr}{\delta} k_b u(x), \end{aligned}$$

which implies (2.23) on (x_1^-, x_1^+) . Continuing this process for all intervals, we complete the proof. $\square$

3 | Characterization of Resonant Frequencies

In this section, we first reformulate the acoustic resonance problem (2.4) (equivalently, Equation (2.17)) as the study of zeros of a trigonometric polynomial (Theorem 3.1). Leveraging properties of zeros of trigonometric polynomials, we then establish a distribution law for the resonances (Theorems 3.3 and 3.5). Finally, we derive the leading-order asymptotic expansions of the resonances k as $\delta \rightarrow 0$ and $\delta \rightarrow \infty$ (Theorem 3.6).

3.1 | Distribution Property

Building on the resonance characterization (2.17), we define the transformed total propagation matrix for the problem (2.4):

$$M_{tot}(k; \sigma) := \frac{(4\sigma)^N}{(1+\sigma)^{2N}} R\left(\frac{1}{\sigma}\right) L(t_{2N-1}k) R(\sigma) L(t_{2N-2}k) \cdots R\left(\frac{1}{\sigma}\right) L(t_1k) R(\sigma), \quad (3.1)$$

where the factor $\frac{(4\sigma)^N}{(1+\sigma)^{2N}}$ is a technical scaling introduced for the subsequent asymptotic analysis. Then, ω is a resonant frequency if and only if the associated wavenumber $k = \omega/v$ satisfies

$$M_{tot}(k; \sigma)_{1,2} \neq 0, \quad M_{tot}(k; \sigma)_{2,2} = 0. \quad (3.2)$$

Moreover, note that $M_{tot}(k, \sigma)$ is an invertible matrix for $\sigma \neq 0$, due to the invertibility of R and L from definition (2.18). It follows directly from (3.2) that a resonance $\omega \neq 0$ is characterized solely by the condition $M_{tot}(k; \sigma)_{2,2} = 0$. By directly expanding the matrix multiplication and computing M_{tot} , we establish the following theorem, which serves as a foundation for the subsequent discussions.

Theorem 3.1. *If $0 \neq \omega = kv$, then ω is a resonant frequency if and only if k is a zero of a analytic function $f(k; \sigma)$ in k defined as*

$$f(k; \sigma) := M_{tot}(k; \sigma)_{2,2} = \sum_{\alpha \in \{-1, 1\}^{2N-1}} (-1)^{\sum_{j=1}^{2N} j \epsilon_{\alpha_j, \alpha_{j-1}}} \left(\frac{1-\sigma}{1+\sigma}\right)^{\sum_{j=1}^{2N} \epsilon_{\alpha_j, \alpha_{j-1}}} e^{i \langle \alpha, t \rangle k}, \quad (3.3)$$

where

$$\alpha = (\alpha_1, \dots, \alpha_{2N-1}) \in \{-1, 1\}^{2N-1}, \quad \alpha_0 = \alpha_{2N} = -1, \quad \epsilon_{jk} = \begin{cases} 1, & j \neq k, \\ 0, & j = k. \end{cases}$$

Example. (i) When $\sigma = 1$ (namely, $\delta = r > 0$), $f(k; \sigma)$ in (3.3) is given by

$$f(k; 1) = e^{-i \|t\|_1 k}.$$

In this case, the scattering problem (2.4) has only the trivial resonance $\omega = 0$, as $e^z \neq 0$ for any $z \in \mathbb{C}$.

(ii) When $N = 1$, there is only one resonator. $f(k; \sigma)$ in (3.3) is given by

$$f(k; \sigma) = e^{-ir\ell_1 k} - \left(\frac{1-\sigma}{1+\sigma} \right)^2 e^{ir\ell_1 k}.$$

It follows that if $\sigma = 1$, there is only the trivial resonance $\omega^* = 0$, and that in the case of $\sigma \neq 1$, all resonances are given by

$$\omega^* = 0, \quad \omega_n = \frac{v}{r\ell_1} \left(n\pi + i \ln \left| \frac{1-\sigma}{1+\sigma} \right| \right), \quad n \in \mathbb{Z}.$$

This implies that as $\delta \rightarrow 0_+$ or $\delta \rightarrow +\infty$, $\omega_n \rightarrow \frac{vn\pi}{r\ell_1}$. Hence, only ω^* and ω_0 are subwavelength resonant frequencies. When $\delta \rightarrow r$, all nontrivial resonant frequencies tend to ∞ (since the imaginary parts of all ω_n uniformly tend to $-\infty$). In other words, there is no nontrivial resonant frequency $\omega(\delta)$ that depends continuously on δ over $(0, +\infty)$.

To proceed, we first recall some fundamental concepts from the theory of entire functions. Consider the following general trigonometric polynomial:

$$P(z) := \sum_{j=1}^n a_j e^{i\lambda_j z}, \quad \text{with } \lambda_1 < \lambda_2 < \dots < \lambda_n \text{ and } 0 \neq a_j \in \mathbb{C}, \quad 1 \leq j \leq n, \quad (3.4)$$

which is a canonical example of both *almost-periodic functions* and *entire functions of exponential type* [25]. For an almost-periodic function g , one can define its mean value $\mathbf{m}(g)$ and Fourier coefficients $a(\lambda)$, respectively, by

$$\mathbf{m}(g) := \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} g(t) dt, \quad a(\lambda) := \mathbf{m}(g(x)e^{-i\lambda x}), \quad \lambda \in \mathbb{R},$$

where the convergence is uniform in $\alpha \in \mathbb{R}$. These coefficients $a(\lambda)$ are nonzero for at most a countable set of λ . This set of λ constitutes the *spectrum* of g , denoted by Λ_g . We refer the reader to [25] for a detailed discussion. The following lemma, taken from [25], plays a crucial role in our analysis.

Lemma 3.2. *Let g be an entire almost-periodic function of exponential type.*

1. *All the zeros of g lie in a horizontal strip parallel to the real axis if and only if the spectrum Λ_g satisfies*

$$\inf \Lambda_g \in \Lambda_g \quad \text{and} \quad \sup \Lambda_g \in \Lambda_g.$$

2. *Let $m_g(x_1, x_2; y_1, y_2)$ denote the number of zeros of g in the rectangle $[x_1, x_2] \times [y_1, y_2]$. If the spectrum Λ_g is bounded, then the linear density of zeros in a horizontal strip:*

$$m_g(y_1, y_2) := \lim_{x_2 - x_1 \rightarrow +\infty} \frac{m_g(x_1, x_2; y_1, y_2)}{x_2 - x_1} \quad (3.5)$$

satisfies

$$\lim_{\substack{y_1 \rightarrow -\infty \\ y_2 \rightarrow +\infty}} m_g(y_1, y_2) = \frac{d}{2\pi},$$

where d is the length of the smallest interval containing the spectrum set Λ_g , or equivalently, $\sup \Lambda_g - \inf \Lambda_g$.

With these concepts in hand, we now return to our main discussion. Example (i) shows that $f(\cdot; 1)$ has no zeros, and thus (2.4) has only a trivial resonant frequency. However, when $\sigma \neq 1$, this is not the case. There are countably many zeros of $f(z; \sigma)$ when $\sigma \neq 1$, which satisfy the following distribution property. In Figure 2 below, we present a numerical simulation to verify the zero density formula (3.6).

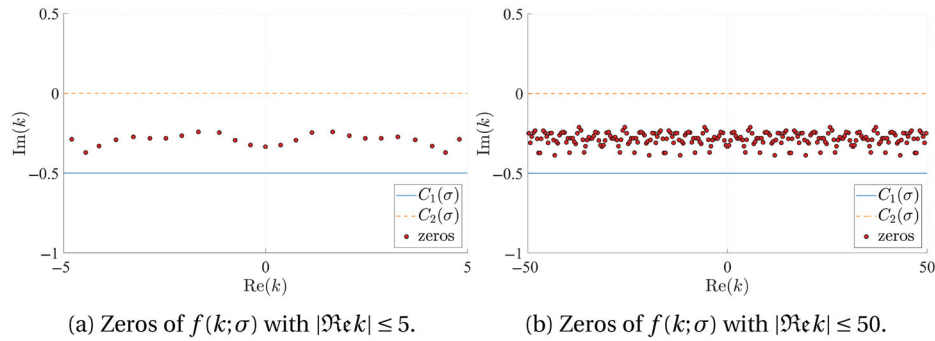


FIGURE 2 | Zeros of $f(k; \sigma)$ for the configuration $\mathbf{t} = (0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4)^T$ and $\sigma = 0.8$. All zeros are confined to the strip $C_1(\sigma) < \Im k < C_2(\sigma)$. In the region $|\Re k| \leq 5$ and $|\Re k| \leq 50$, the argument principle yields 25 zeros and 245 zeros, giving a density of 2.5 and 2.45 zeros per unit length, respectively. These empirical densities align closely with the theoretical one $\|\mathbf{t}\|_1/\pi \approx 2.451$ established in Theorem 3.3.

Theorem 3.3. For fixed $\sigma \neq 1$, there exist constants $C_1(\sigma), C_2(\sigma) \in \mathbb{R}$ such that all zeros of $f(k; \sigma)$ satisfy

$$C_1(\sigma) < \Im k < C_2(\sigma).$$

If $\delta \in \mathbb{R}_+$, Theorem 2.2 (2) allows us to set $C_2(\sigma) = 0$. Moreover, for any $x_1 < x_2$, we define the number of zeros of $f(\cdot; \sigma)$ in the rectangle $[x_1, x_2] \times [C_1(\sigma), C_2(\sigma)]$:

$$n(x_1, x_2) = \#\left\{k \in \mathbb{C} \mid f(k; \sigma) = 0, x_1 < \Re k < x_2, C_1(\sigma) < \Im k < C_2(\sigma)\right\}.$$

Then, we have

$$\lim_{x_2 - x_1 \rightarrow +\infty} \frac{n(x_1, x_2)}{x_2 - x_1} = \frac{\|\mathbf{t}\|_1}{\pi}. \quad (3.6)$$

In particular, $f(\cdot; \sigma)$ possesses countably many zeros and consequently the scattering problem (2.4) admits countably many resonant frequencies.

Proof. When $\sigma \neq 1$, the expansion (3.3) implies that at least two terms of $f(k; \sigma)$ are non-zero. Specifically, $f(k; \sigma)$ contains the terms $e^{-i\|\mathbf{t}\|_1 k}$ and $-\left(\frac{\sigma-1}{\sigma+1}\right)^2 e^{i\|\mathbf{t}\|_1 k}$, corresponding to the choices $\alpha = \mathbf{1}$ and $\alpha = -\mathbf{1}$, respectively. Furthermore, for any $\alpha \in \{-1, 1\}^{2N-1}$ with $\alpha \neq \pm \mathbf{1}$, we have

$$-\|\mathbf{t}\|_1 < \langle \alpha, \mathbf{t} \rangle < \|\mathbf{t}\|_1.$$

Thus we can rewrite f in the form of (3.4) with $\lambda_1 = -\|\mathbf{t}\|_1$ and $\lambda_n = \|\mathbf{t}\|_1$. The Fourier coefficients of f can be computed as follows:

$$a(\lambda) = \frac{1}{2T} \sum_{j=1}^n a_j \lim_{T \rightarrow +\infty} \int_{-T+\alpha}^{T+\alpha} e^{i(\lambda_j - \lambda)t} dt = \begin{cases} a_j, & \lambda = \lambda_j \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the spectrum of f is given by $\Lambda_f = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

By Lemma 3.2(1), there exist constants $C_1(\sigma) < C_2(\sigma)$ such that all zeros of f lie within the strip $C_1(\sigma) < \Im k < C_2(\sigma)$. Let $m_f(y_1, y_2)$ be defined as in (3.5). Then, we have $m_f(y_1, y_2) = m_f(C_1(\sigma), C_2(\sigma))$ for all $y_1 \leq C_1(\sigma)$ and $y_2 \geq C_2(\sigma)$. Applying Lemma 3.2(2), we obtain

$$m_f(C_1(\sigma), C_2(\sigma)) = \lim_{\substack{y_1 \rightarrow -\infty \\ y_2 \rightarrow +\infty}} m_f(y_1, y_2) = \frac{\lambda_n - \lambda_1}{2\pi} = \frac{\|\mathbf{t}\|_1}{\pi}.$$

Consequently, we have (3.6) by

$$\lim_{x_2-x_1 \rightarrow +\infty} \frac{n(x_1, x_2)}{x_2 - x_1} = \lim_{x_2-x_1 \rightarrow +\infty} \frac{m_f(x_1, x_2; C_1(\sigma), C_2(\sigma))}{x_2 - x_1} = m_f(C_1(\sigma), C_2(\sigma)). \quad \square$$

Remark. Theorem 3.3 reveals a fundamental distinction between the one-dimensional and three-dimensional problems. In three dimensions, the bounds on the imaginary part of the resonant frequencies generally depend on its real part and no single (uniform) line can bound all resonances from below; see [26] for the resonance-free region. In the one-dimensional case, however, we prove the existence of such a uniform lower line for all the resonant frequencies. This result also identifies a so-called resonance-free region, which is of great theoretical interest [27, 28].

3.2 | Limiting Cases of $\delta \rightarrow 0$ and $\delta \rightarrow \infty$

We will characterize the limiting distribution of resonant frequencies of (2.4) when $\delta \rightarrow 0$ and $\delta \rightarrow \infty$.

From definition (2.18), we first obtain

$$\lim_{\sigma \rightarrow 0} \frac{2}{1+\sigma} R(\sigma) = R_+ := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \lim_{\sigma \rightarrow 0} \frac{2\sigma}{1+\sigma} R\left(\frac{1}{\sigma}\right) = R_- := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (3.7)$$

Thus, as $\sigma \rightarrow 0$, the matrix $M_{tot}(k; \sigma)$ in (3.1) uniformly converges to

$$\begin{aligned} M_{tot}(k; 0) &:= R_- L(t_{2N-1}k) R_+ L(t_{2N-2}k) R_- \cdots R_- L(t_1k) R_+ \\ &= (2i)^{2N-1} \prod_{j=1}^{2N-1} \sin(t_jk) \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \end{aligned}$$

on any compact set $K \subset \mathbb{C}$. We then have the following lemma.

Lemma 3.4. When $\sigma = 0$, the analytic function $f(k; 0)$ in (3.3) becomes

$$f(k; 0) = -(2i)^{2N-1} \prod_{j=1}^{2N-1} \sin(t_jk).$$

Then, $k \in \mathbb{C}$ is a zero of $f(\cdot, 0)$ if and only if $k \in E := \cup_{j=1}^{2N-1} (\pi\mathbb{Z}/t_j)$. When $k \in E$, k is a zero of order

$$n(k) := \#\left\{j \mid t_jk \in \pi\mathbb{Z}, 1 \leq j \leq 2N-1\right\}.$$

In particular, all zeros of $f(\cdot; 0)$ are real and 0 is a zero of order $2N-1$.

Since $f(k; \sigma)$ uniformly converges to $f(k; 0)$ on any compact set as $\sigma \rightarrow 0$, the Rouché's Theorem implies that there are exactly $n(k)$ zeros of $f(\cdot; \sigma)$ in a small neighborhood of k for every $k \in E$ when $|\sigma|$ is small enough. This shows that (2.4) has exactly $n(k)$ nontrivial resonant frequencies near kv for every $k \in E$ when $\delta \rightarrow 0$. Moreover, by the Newton polygon method as in the next section, we can deduce that $\omega(\delta)$ is an analytic function of δ^s for certain rational s satisfying $s \geq \frac{1}{n(k)}$. According to Theorem 2.2, similar conclusions hold as $\delta \rightarrow \infty$. We summarize our discussion as follows. Figure 3 provides a numerical illustration of Theorem 3.5.

Theorem 3.5. For δ small enough (or large enough), there exist exactly $n(k^*)$ (counting multiplicities) nontrivial resonant frequencies $\omega = kv$ near k^*v , where $k^* \in \cup_{j=1}^{2N-1} \pi\mathbb{Z}/t_j$, with $C_1(\sigma) < \Im mk < C_2(\sigma)$ for $C_1(\sigma), C_2(\sigma)$ given as in Theorem 3.3. In particular, near each k^*v , the resonance $\omega(\delta)$ is an analytic function of δ^s (or δ^{-s} , respectively) for certain rational $s \geq \frac{1}{n(k^*)}$.

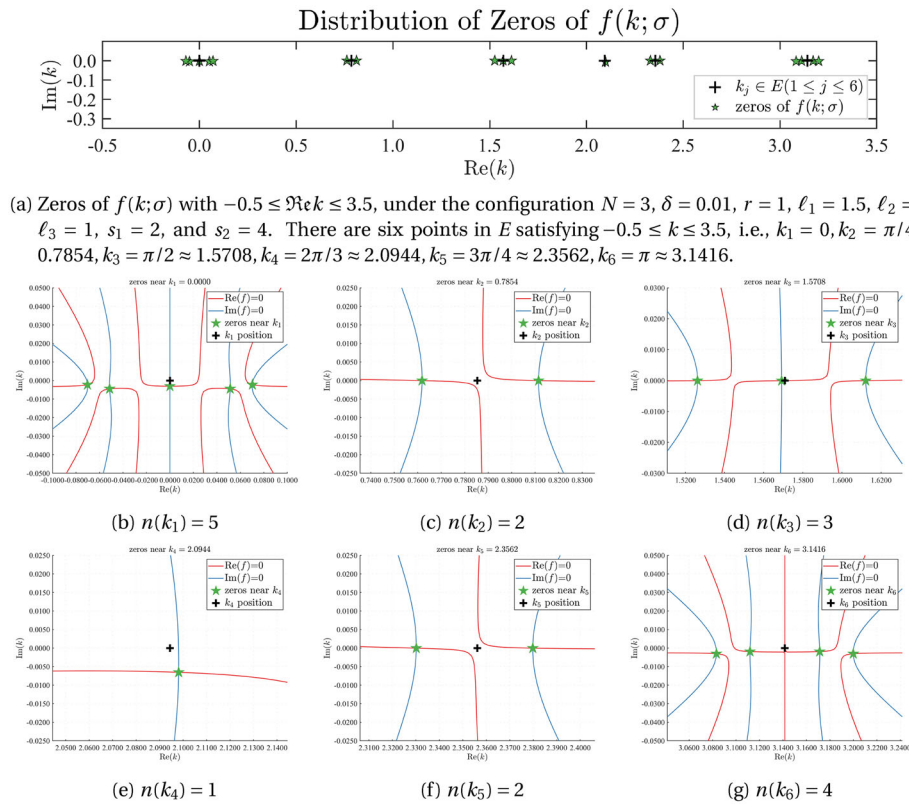


FIGURE 3 | Zeros of $f(k; \sigma)$ with sufficiently small δ . Panels (b)–(g): Zoom-in views near k_1 – k_6 with different orders $n(k_i)$. We see that exactly $n(k)$ zeros are located near each $k \in E$, where the set E is given in Lemma 3.4.

3.3 | Asymptotic Expansions of Resonances

We next analyze the asymptotic behaviors of resonances as $\delta \rightarrow 0$ and $\delta \rightarrow \infty$. For simplicity, we focus on the case where $n(k) = 1$. Notably, this already covers the case in the concurrent work [26], where asymptotic expansions of resonances are derived for a single resonator in three dimensions. Compared to their results, our asymptotic formulas for the one-dimensional case are more explicit. For cases where $n(k_0) > 1$, asymptotic analysis can in principle be carried out using the Newton polygon method, as presented in the next section for subwavelength resonances. However, such computations are operationally cumbersome and will be reported in a forthcoming work.

For notational simplicity, we will sometimes use $L_j(k)$ to denote $L(t_j k)$. Let

$$\nu := \frac{2\sigma}{1+\sigma}, \quad R := \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.8)$$

For the components $\frac{4\sigma}{(1+\sigma)^2} R \left(\frac{1}{\sigma} \right) L(k) R(\sigma)$ in $M_{tot}(k; \sigma)$, we observe that

$$\frac{4\sigma}{(1+\sigma)^2} P R \left(\frac{1}{\sigma} \right) L(k) R(\sigma) P = (R + \nu S) L(k) (R + \nu S), \quad \text{with } P = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

Therefore, it holds that

$$P M_{tot}(k; \sigma) P = G(k; \nu), \quad f(k; \sigma) = g(k; \nu),$$

where $G(k; \nu)$ and $g(k; \nu)$ are defined by

$$G(k; \nu) := (R + \nu S) L_{2N-1}(k) (R + \nu S) L_{2N-2}(k) (R + \nu S) \cdots L_1(k) (R + \nu S), \quad (3.9)$$

$$g(k; \nu) := G(k, \nu)_{2,2}. \quad (3.10)$$

For the case when $\delta \rightarrow 0$, from (3.9), we can expand $G(k; \nu)$ and $g(k; \nu)$ as

$$\begin{cases} G(k; \nu) := G_0(k) + G_1(k)\nu + G_2(k)\nu^2 + \cdots + G_{2N}(k)\nu^{2N}, \\ g(k; \nu) := g_0(k) + g_1(k)\nu + g_2(k)\nu^2 + \cdots + g_{2N}(k)\nu^{2N}. \end{cases} \quad (3.11)$$

Using the following easily verifiable identities, with $R_{\pm}$, R , and S defined in (3.7) and (3.8),

$$\begin{aligned} RL_j(k)R &= -2i \sin(t_j k)R, & RL_{j+1}(k)SL_j(k)R &= 2 \cos((t_{j+1} - t_j)k)R, \\ SL_{2N-1}(k)R &= L_{2N-1}(-k)R_-, & RL_1(k)S &= R_+L_1(-k), \end{aligned} \quad (3.12)$$

we can explicitly compute $G_1(k)$ in the expansion (3.11) as

$$\begin{aligned} G_1(k) &= (2i)^{2N-2} \prod_{j=1}^{2N-2} \sin(t_j k) L(-t_{2N-1} k) R_- + (2i)^{2N-2} \prod_{j=2}^{2N-1} \sin(t_j k) R_+ L(-t_1 k) \\ &\quad - 2(2i)^{2N-3} \sum_{j=1}^{2N-2} \cos((t_{j+1} - t_j)k) \prod_{\substack{s=1 \\ s \neq j, j+1}}^{2N-1} \sin(t_s k) R. \end{aligned} \quad (3.13)$$

We are now in a position to derive the asymptotic expansions of the resonant frequencies for wavenumbers k_0 satisfying $n(k_0) = 1$. Figure 4 provides a numerical illustration of Theorem 3.6.

Theorem 3.6. Assume $n(k_0) = 1$. When $\delta \rightarrow 0$ (or $\delta \rightarrow \infty$), the scattering problem (2.4) has a unique resonance $\omega(\delta)$ in a neighborhood of $k_0 v$, which is an analytic function of δ (or δ^{-1} , respectively) with the first-order asymptotics:

$$\omega(\delta) = k_0 v + \begin{cases} \frac{\cot(s_1 k_0) - i}{r^2 \ell_1} v \delta + O(\delta^2), & \pi \mid r \ell_1 k_0, \\ \frac{\cot(s_{N-1} k_0) - i}{r^2 \ell_N} v \delta + O(\delta^2), & \pi \mid r \ell_N k_0, \\ \frac{\cot(s_{j-1} k_0) + \cot(s_j k_0)}{r^2 \ell_j} v \delta + O(\delta^2), & \pi \mid r \ell_j k_0, \ 1 < j < N, \\ \frac{\cot(r \ell_j k_0) + \cot(r \ell_{j+1} k_0)}{r s_j} v \delta + O(\delta^2), & \pi \mid s_j k_0, \ 1 \leq j \leq N-1, \end{cases} \quad (3.14)$$

as $\delta \rightarrow 0$, and as $\delta \rightarrow \infty$,

$$\omega(\delta) = k_0 v + \begin{cases} \frac{\cot(s_1 k_0) - i}{\ell_1} \frac{v}{\delta} + O(\delta^{-2}), & \pi \mid r \ell_1 k_0, \\ \frac{\cot(s_{N-1} k_0) - i}{\ell_N} \frac{v}{\delta} + O(\delta^{-2}), & \pi \mid r \ell_N k_0, \\ \frac{\cot(s_{j-1} k_0) + \cot(s_j k_0)}{\ell_j} \frac{v}{\delta} + O(\delta^{-2}), & \pi \mid r \ell_j k_0, \ 1 < j < N, \\ \frac{\cot(r \ell_j k_0) + \cot(r \ell_{j+1} k_0)}{s_j} \frac{v r}{\delta} + O(\delta^{-2}), & \pi \mid s_j k_0, \ 1 \leq j \leq N-1. \end{cases}$$

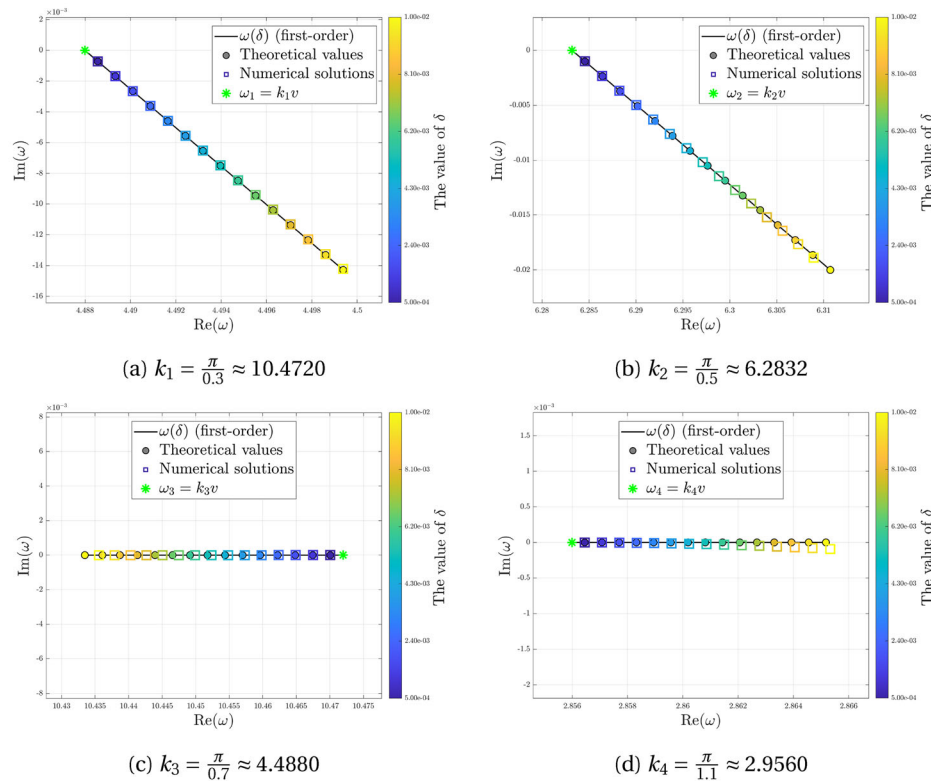


FIGURE 4 | Asymptotic behavior of $\omega(\delta)$ for sufficiently small δ is analyzed near $\omega_j = k_j v$ with $n(k_j) = 1$ for $1 \leq j \leq 4$, under the configuration $N = 3$, $r = 1$, $v = 1$, $\ell_1 = 0.7$, $\ell_2 = 0.3$, $\ell_3 = 0.5$, $s_1 = 0.2$, and $s_2 = 1.1$. Panels (a)–(d) correspond to the first through fourth cases in (3.14), respectively.

Proof. We will focus on proving the case $\delta \rightarrow 0$; the case $\delta \rightarrow \infty$ then follows straightforwardly from Theorem 2.2. Moreover, noting that $\nu = \frac{2\sigma}{1+\sigma} = \frac{2\delta/r}{1+\delta/r}$ and the relation $\omega = kv$, it suffices to derive the expansion of $k(\nu)$ in terms of ν .

According to Lemma 3.4, we have

$$g(k; 0) = -(2i)^{2N-1} \prod_{j=1}^{2N-1} \sin(t_j k).$$

$n(k_0) = 1$ implies that there exists an $n \in \mathbb{Z}$ and $1 \leq j \leq 2N - 1$ such that $t_j k_0 = n\pi$, while $\pi \nmid t_s k_0$ for any $s \neq j$. Thus, we derive

$$g(k_0; 0) = 0, \quad \frac{\partial g}{\partial k}(k_0; 0) = (-1)^{n+1} (2i)^{2N-1} t_j \prod_{\substack{s=1 \\ s \neq j}}^{2N-1} \sin(t_s k_0) \neq 0.$$

By the implicit function theorem, there exists a unique function $k(\nu)$ defined in a neighborhood of $\nu = 0$, which is analytic in ν and satisfies $g(k(\nu); \nu) = 0$ and $k(0) = k_0$. Moreover, since

$$\frac{\partial g}{\partial \nu}(k; 0) = g_1(k) = G_1(k)_{2,2},$$

the expansion (3.13) gives

$$\frac{\partial g}{\partial \nu}(k_0; 0) = (-1)^{n+1} 2(2i)^{2N-3} \cdot \begin{cases} e^{-it_2 k_0} \prod_{s=3}^{2N-1} \sin(t_s k_0), & j = 1, \\ \sin((t_{j-1} + t_{j+1})k_0) \prod_{\substack{s=1 \\ |s-j|>1}}^{2N-1} \sin(t_s k_0), & 1 < j < 2N-1, \\ e^{-it_{2N-2} k_0} \prod_{s=1}^{2N-3} \sin(t_s k_0), & j = 2N-1. \end{cases}$$

It follows that

$$k'(0) = -\frac{\frac{\partial g}{\partial \nu}(k_0; 0)}{\frac{\partial g}{\partial k}(k_0; 0)} = \begin{cases} \frac{\cot(t_2 k_0) - i}{2t_1}, & j = 1, \\ \frac{\cot(t_{j-1} k_0) + \cot(t_{j+1} k_0)}{2t_j}, & 1 < j < 2N-1, \\ \frac{\cot(t_{2N-2} k_0) - i}{2t_{2N-1}}, & j = 2N-1. \end{cases}$$

Substituting this into the expansion $k(\nu) = k_0 + k'(0)\nu + O(\nu^2)$, along with the definition of the vector $\mathbf{t}$ in (2.15), completes the proof. $\square$

4 | Subwavelength Resonances and Capacitance Matrix Theory

In this section, we study the resonant frequencies in the subwavelength regime; see Definition 4.1. Specifically, we recover the capacitance matrix theory in [15] for subwavelength resonances using the propagation matrix approach, revealing the relationships between the two frameworks. Our new method is conceptually distinct from the variational approach in [15, 16], which is based on Dirichlet-to-Neumann maps, and is more natural for the one-dimensional problem.

Definition 4.1. We call $\omega(\delta) \in \mathbb{C}$ a *subwavelength resonant frequency* if $\omega(\delta)$ depends continuously on $\delta \in \mathbb{C}$ for sufficiently small (or large) δ and satisfies $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ (or $\delta \rightarrow \infty$).

4.1 | Subwavelength Eigenfrequencies

In this section, we analyze the asymptotic behavior of the subwavelength resonant frequencies. Unlike the expansion in the previous section, the implicit function theorem is no longer applicable here because zero is a high-order root of the characteristic function $f(k; \sigma)$ in (3.3) (or equivalently, $g(k; \nu)$ in (3.11)). Instead, we employ the Newton polygon method from multivariate complex analysis to obtain the asymptotic expansion (see Appendix A).

Let

$$\theta_j = \frac{1}{t_j t_{j+1}}, \quad j = 0, 1, \dots, 2N-1, \quad (4.1)$$

with $t_0 = t_{2N} = 1$. We first introduce the expansion of $g_l(k)$ in (3.11) in Proposition 4.2 below, whose proof is deferred to Appendix B for ease of exposition.

Proposition 4.2. When $k \rightarrow 0$, for $1 \leq l \leq N-1$, we have

$$g_l(k) = \left(\prod_{j=1}^{2N-1} t_j \right) \left[2^l (-2ik)^{2N-1-2l} \sum_{1 \leq j_1 < j_2 < \dots < j_l \leq 2N-2} \left(\prod_{m=1}^l \theta_{j_m} \right) \right]$$

$$\begin{aligned}
 & + \theta_0 2^{l-1} (-2ik)^{2N-2l} \sum_{2 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-2} \left(\prod_{m=1}^l \theta_{j_m} \right) \\
 & + \theta_{2N-1} 2^{l-1} (-2ik)^{2N-2l} \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-3} \left(\prod_{m=1}^l \theta_{j_m} \right) + O(k^{2N-2l+1}) \Bigg], \quad (4.2)
 \end{aligned}$$

with $i < j$ meaning $j - i > 1$, and for $l = 0, N$,

$$g_0(k) = \left(\prod_{j=1}^{2N-1} t_j \right) (-2ki)^{2N-1} + O(k^{2N+1}), \quad g_N(k) = 2^N + O(k).$$

Leveraging these asymptotic expansions, we can recover the capacitance matrix theory in [15] from the propagation matrix method. Recall from [15] that the generalized capacitance matrix is defined as: with $\theta_j = \frac{1}{t_j t_{j+1}}$ in (4.1),

$$C := \begin{pmatrix} \theta_1 & -\theta_1 & & & & & \\ -\theta_2 & \theta_2 + \theta_3 & -\theta_3 & & & & \\ & -\theta_4 & \theta_4 + \theta_5 & -\theta_5 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\theta_{2N-4} & \theta_{2N-4} + \theta_{2N-3} & -\theta_{2N-3} & \\ & & & & -\theta_{2N-2} & \theta_{2N-2} & \end{pmatrix}. \quad (4.3)$$

We now introduce several auxiliary lemmas.

Lemma 4.3. For the generalized capacitance matrix C , we have

1. It has N different eigenvalues $0 = \lambda_1 < \lambda_2 < \dots < \lambda_N$.
2. Let P_N be the determinant of $C - \lambda I$, and Q_N^i the (i, i) -cofactor for $i = 1, N$. Then

$$\begin{aligned}
 P_N(\lambda, \theta_1, \theta_2, \dots, \theta_{2N-2}) &= |C - \lambda I| = \sum_{l=0}^{N-1} (-\lambda)^{N-l} \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_l \leq 2N-2}} \prod_{m=1}^l \theta_{j_m}, \quad (4.4) \\
 Q_N^1(\lambda, \theta_2, \theta_3, \dots, \theta_{2N-2}) &= [C - \lambda I]_{11} = \sum_{l=1}^N (-\lambda)^{N-l} \sum_{\substack{2 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-2}} \prod_{m=1}^{l-1} \theta_{j_m}, \\
 Q_N^N(\lambda, \theta_1, \theta_2, \dots, \theta_{2N-3}) &= [C - \lambda I]_{NN} = \sum_{l=1}^N (-\lambda)^{N-l} \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-3}} \prod_{m=1}^{l-1} \theta_{j_m}.
 \end{aligned}$$

3. Suppose that $\lambda \neq 0$ is an eigenvalue of C , and $\mathbf{a} = (a_1, a_2, \dots, a_N)^\top$, $\mathbf{b} = (b_1, b_2, \dots, b_N)$ are the corresponding right and left eigenvectors. Then

$$\frac{a_i b_i}{\sum_{j=1}^N a_j b_j} = \frac{Q_N^i(\lambda)}{P'_N(\lambda)}, \quad i = 1, N.$$

Proof. The property (1) follows from the fact that C is a tridiagonal matrix with nonzero off-diagonal elements; see [29, Lemma 7.7.1].

For property (2), we prove it by mathematical induction. First, $P_1(\lambda, \theta_1) = -\lambda + \theta_1$. Then by adding the last column of $C - \lambda I$ to its second-to-last column and expanding along the last column, we obtain the following recurrence formula:

$$P_N(\lambda; \theta_1, \theta_2, \dots, \theta_{2N-2}) = (\theta_{2N-2} - \lambda)P_{N-1}(\lambda; \theta_1, \theta_2, \dots, \theta_{2N-4}) \\ + \theta_{2N-3}P_{N-1}(\lambda; \theta_1, \theta_2, \dots, \theta_{2N-5}, 0).$$

Using mathematical induction, we derive the expansion (4.4) of $P_N(\lambda; \theta_1, \dots, \theta_{2N-2})$. Similarly, the expansions of Q_N^1 and Q_N^N can also be obtained.

For property (3), since λ is a simple eigenvalue of C , the matrix $C - \lambda I$ has rank $N - 1$. Then, the adjugate matrix of $C - \lambda I$ is a rank-1 matrix, and there exists a $0 \neq \beta \in \mathbb{C}$ such that $\text{adj}(C - \lambda I) = \beta \mathbf{a}\mathbf{b}$. Thus, we have

$$Q_N^i(\lambda) = [C - \lambda I]_{ii} = \beta a_i b_i, \quad i = 1, N, \quad P_N'(\lambda) = \text{tr}(\text{adj}(C - \lambda I)) = \beta \sum_{j=1}^N a_j b_j.$$

Combining these equations completes the proof. $\square$

Remark. Note that we can express the generalized capacitance matrix in (4.3) as $C = V^{-1}C$, where C is a symmetric matrix given by

$$C := \begin{pmatrix} \frac{1}{t_2} & -\frac{1}{t_2} & & & & \\ -\frac{1}{t_2} & \frac{1}{t_2} + \frac{1}{t_4} & -\frac{1}{t_4} & & & \\ & -\frac{1}{t_4} & \frac{1}{t_4} + \frac{1}{t_6} & -\frac{1}{t_6} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{t_{2N-4}} & \frac{1}{t_{2N-4}} + \frac{1}{t_{2N-2}} & -\frac{1}{t_{2N-2}} \\ & & & & -\frac{1}{t_{2N-2}} & \frac{1}{t_{2N-2}} \end{pmatrix},$$

and $V := \text{diag}(t_1, t_3, \dots, t_{2N-1})$. This decomposition implies a direct relation between the right and left eigenvectors. Specifically, if $\mathbf{a} = (a_1, a_2, \dots, a_N)^\top$ and $\mathbf{b} = (b_1, b_2, \dots, b_N)$ are the right and left eigenvectors of C associated with the same eigenvalue, respectively, then there holds

$$\mathbf{b}^\top = V\mathbf{a}, \quad \text{equivalently,} \quad b_j = t_{2j-1}a_j, \quad \text{for } j = 1, 2, \dots, N.$$

We now present the asymptotic expansions of the scattering resonance $\omega(\delta)$ in the subwavelength regime. In what follows, $\sqrt{z}$ is defined on $\mathbb{C} \setminus [0, +\infty)$ with the branch $\Im \mathbf{m} \sqrt{z} > 0$. For $z \in [0, +\infty)$, we define $\sqrt{z} := \lim_{\varepsilon \rightarrow 0^+} \sqrt{z + \varepsilon i}$. We remark that the following result can be generalized to the case where the v_b 's differ across the D_i 's.

Theorem 4.4. *Let $\delta \in \mathbb{C}$. In the cases where $\delta \rightarrow 0$ (or $\delta \rightarrow \infty$), the scattering problem (2.4) exhibits exactly $2N$ subwavelength resonant frequencies:*

- A trivial frequency $\omega^* = 0$.
- The first nontrivial eigenfrequency $\omega_1(\delta)$ is an analytic function of δ (or δ^{-1}), with its leading asymptotic expansion given by

$$\omega_1(\delta) = -2i\delta \frac{v}{r^2 \sum_{j=1}^N \ell_j} + O(\delta^2), \quad \delta \rightarrow 0, \\ \text{or } \omega_1(\delta) = -2i \frac{1}{\delta} \frac{v}{\sum_{j=1}^N \ell_j} + O(\delta^{-2}), \quad \delta \rightarrow \infty.$$

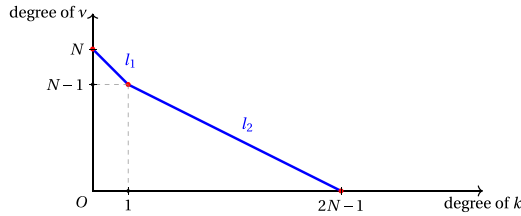


FIGURE 5 | The lower boundary of the convex hull.

- The remaining $2N - 2$ frequencies are analytic functions of $\delta^{\frac{1}{2}}$ (or $\delta^{-\frac{1}{2}}$), and their leading-order asymptotic expansion is given by

$$\omega_i^{\pm}(\delta) = \pm v \sqrt{\frac{\lambda_i \delta}{r}} - i \delta \frac{v}{2r^2} \frac{a_{i1}^2 + a_{iN}^2}{\sum_{j=1}^N a_{ij}^2 \ell_j} + O(\delta^{\frac{3}{2}}), \quad \delta \rightarrow 0, \quad 2 \leq i \leq N,$$

$$\text{or } \omega_i^{\pm}(\delta) = \pm v \sqrt{\frac{\lambda_i r}{\delta}} - i \frac{1}{\delta} \frac{v}{2} \frac{a_{i1}^2 + a_{iN}^2}{\sum_{j=1}^N a_{ij}^2 \ell_j} + O(\delta^{-\frac{3}{2}}), \quad \delta \rightarrow \infty, \quad 2 \leq i \leq N,$$

where $0 = \lambda_1 < \lambda_2 < \dots < \lambda_N$ are eigenvalues of the matrix C (4.3) and $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{iN})^T$ is the associated eigenvectors.

Proof. Similar to the proof of Theorem 3.6, it suffices to consider the case $\delta \rightarrow 0$ and derive the asymptotics of the wavenumber $k(\nu)$ in terms of $\nu = \frac{2\delta/r}{1+\delta/r}$.

First, we apply the Newton polygon method reviewed in Appendix A to determine the leading-order asymptotics of $k(\nu)$. Specifically, by (3.11), we expand $g(k; \nu)$ around $k = 0$ as

$$g(k; \nu) = \sum_{l=1}^{2N} g_l(k) \nu^l = \sum_{l=1}^{2N} \sum_{j=2N-1-2l}^{\infty} c_{jl} k^j \nu^l.$$

Here, c_{jl} can be determined by $g_l(k)$ in Proposition 4.2. The associated Newton polygon can be plotted in the $\mathbb{R}^2$ plane, where the lower boundary of the convex hull comprises two connected piecewise linear segments, l_1 and l_2 , with slopes -1 and $-\frac{1}{2}$, respectively, as shown in Figure 5.

The sets of points on the edges l_1 and l_2 are defined as follows:

$$E_1 := l_1 \cap \mathbb{Z}^2 = \{(0, N), (1, N-1)\},$$

$$E_2 := l_2 \cap \mathbb{Z}^2 = \{(1, N-1), (3, N-2), (5, N-3), \dots, (2N-1, 0)\}.$$

For the first edge l_1 , by (A.2), the asymptotic form is $k \sim c\nu$ with $c \neq 0$, and the balance equation is

$$\sum_{(j,l) \in E_1} c_{jl} c^j = 0,$$

where, using (4.2), we have

$$c_{0N} = 2^N, \quad c_{1,N-1} = -2^N i \sum_{j=1}^N t_{2j-1}.$$

Recall that $l_j = t_{2j-1}$ for $j = 1, \dots, N$. Solving the balance equation gives $c^{-1} = i \sum_{j=1}^N t_{2j-1}$, leading to the asymptotic form of the zero:

$$k_1(\nu) \sim -i \frac{\nu}{\sum_{j=1}^N t_{2j-1}}, \quad \nu \rightarrow 0.$$

Similarly, for the second edge l_2 , the asymptotic form is $k \sim c\sqrt{\nu}$ with $c \neq 0$, and the balance equation is given by

$$\sum_{(j,l) \in E_2} c_{jl} c^j = 0, \quad (4.5)$$

where, according to (4.2),

$$c_{2(N-l)-1,l} = 2^l (-2i)^{2(N-l)-1} \prod_{j=1}^{2N-1} t_j \sum_{\substack{1 \leq j_1 < j_2 < \dots < j_l \leq 2N-1}} \prod_{m=1}^l \theta_{j_m}, \quad 0 \leq l \leq N-1.$$

Using the characteristic polynomial $P_N(z)$ from Lemma 4.3, we have

$$0 = \sum_{(j,l) \in E_2} c_{jl} c^j = i 2^{N-1} \left(\prod_{j=1}^{2N-1} t_j \right) \frac{P_N(2c^2)}{c}.$$

It follows that the solutions to Equation (4.5) can be characterized by the nonzero eigenvalues λ_j , $2 \leq j \leq N$, of C :

$$c_i^\pm = \pm \sqrt{\frac{\lambda_i}{2}}, \quad 2 \leq i \leq N.$$

Consequently, the other $2N - 2$ zeros have the asymptotic form

$$k_i^\pm(\nu) \sim \pm \sqrt{\frac{1}{2} \lambda_i} \nu, \quad \nu \rightarrow 0, \quad 2 \leq i \leq N.$$

Next, we compute the second-order correction terms. Define

$$h_i^\pm(\alpha; \nu^{\frac{1}{2}}) := \nu^{-N+\frac{1}{2}} g\left(\nu^{\frac{1}{2}} \left(\pm \sqrt{\frac{1}{2} \lambda_i} + \alpha\right), \nu\right), \quad 2 \leq i \leq N,$$

with g defined in (3.10). A direct computation gives

$$\begin{aligned} h_i^\pm(\alpha; \nu^{\frac{1}{2}}) &= \nu^{\frac{1}{2}} \sum_{l=1}^N c_{2N-2l,l} \left(\pm \sqrt{\frac{1}{2} \lambda_i}\right)^{2N-2l} + \alpha \sum_{l=0}^{N-1} (2N-2l-1) c_{2N-2l-1,l} \left(\pm \sqrt{\frac{1}{2} \lambda_i}\right)^{2N-2l-2} + \sum_{\substack{j,l \in \mathbb{Z} \\ 2j+l \geq 2}} d_{jl} \nu^j \alpha^l \\ &= \left(\prod_{j=1}^{2N-1} t_j \right) \left[\nu^{\frac{1}{2}} 2^{N-1} (\theta_0 Q_N^1(\lambda_i) + \theta_{2N-1} Q_N^N(\lambda_i)) - \alpha i 2^{N+1} P'_N(\lambda_i) \right] + \sum_{\substack{j,l \in \mathbb{Z} \\ 2j+l \geq 2}} d_{jl}^\pm \nu^j \alpha^l. \end{aligned} \quad (4.6)$$

Notice that $h_i^\pm(\alpha; \nu^{\frac{1}{2}})$ is analytic in α and $\nu^{\frac{1}{2}}$, with $h_i^\pm(0; 0) = 0$, and satisfies

$$\begin{aligned} \frac{\partial}{\partial \alpha} h_i^\pm(0; 0) &= -2^{N+1} i \left(\prod_{j=1}^{2N-1} t_j \right) P'_N(\lambda_i), \\ \frac{\partial}{\partial \nu^{\frac{1}{2}}} h_i^\pm(0; 0) &= 2^{N-1} \left(\prod_{j=1}^{2N-1} t_j \right) (\theta_0 Q_N^1(\lambda_i) + \theta_{2N-1} Q_N^N(\lambda_i)). \end{aligned}$$

Since the eigenvalue λ_i has multiplicity one for C , it follows that $P'_N(\lambda_i) \neq 0$. By the implicit function theorem, there exists a unique analytic function $\alpha_i^\pm(\nu^{\frac{1}{2}})$ such that

$$h_i^\pm\left(\alpha_i^\pm(\nu^{\frac{1}{2}}); \nu^{\frac{1}{2}}\right) = 0, \quad 2 \leq i \leq N,$$

with $\alpha_i^\pm(0) = 0$, for ν sufficiently close to zero. Moreover, using the implicit function theorem method for the expansion (4.6), and combining Lemma 4.3, we have

$$\alpha_i^\pm(\nu^{\frac{1}{2}}) = -\nu^{\frac{1}{2}} \frac{i}{4} \frac{a_{i1}^2 + a_{iN}^2}{\sum_{j=1}^N a_{ij}^2 t_{2j-1}} + O(\nu), \quad 2 \leq i \leq n.$$

Thus, $k_i^\pm(\nu) = \nu^{\frac{1}{2}} \left(\pm \sqrt{\frac{1}{2}\lambda_i} + \alpha_i^\pm(\nu^{\frac{1}{2}}) \right)$, $2 \leq i \leq N$, are the $2N - 2$ zeros of $g(k; \nu)$, which are analytic functions of $\nu^{\frac{1}{2}}$. Their leading-order asymptotic expansions are given by

$$k_i^\pm(\nu) = \pm \sqrt{\frac{\lambda_i \nu}{2}} - i \frac{\nu}{4} \frac{a_{i1}^2 + a_{iN}^2}{\sum_{j=1}^N a_{ij}^2 t_{2j-1}} + O(\nu^{\frac{3}{2}}), \quad i = 2, 3, \dots, N.$$

Finally, for $k_1(\nu)$, define

$$h_1(\alpha, \nu) = \nu^{-N} g\left(\nu \left(i \frac{\nu}{\sum_{j=1}^N t_{2j-1}} + \alpha \right); \nu\right).$$

We find that $h_1(0, 0) = 0$ and $\frac{\partial}{\partial \alpha} h_1(0, 0) = c_{1,N-1} \neq 0$. By the implicit function theorem, it follows that $k_1(\nu)$ is an analytic function of ν . $\square$

4.2 | Eigenmodes of the Scattering Problem

In this subsection, we characterize the eigenmodes of (2.4) corresponding to the $2N - 1$ nontrivial subwavelength resonant frequencies. The following Theorem 4.5 treats the case $\delta \rightarrow 0$; the result for $\delta \rightarrow \infty$ follows from Theorem 2.2 (see Corollary 4.6).

Theorem 4.5. Suppose that $\omega(\delta) = k(\delta)\nu = O(\delta^{1/2})$ is a subwavelength resonant frequency for the scattering problem (2.4) in case $\delta \rightarrow 0$ and analytic about $\delta^{1/2}$, $u(x)$ is the corresponding nontrivial solution with $u(x) = e^{-i\frac{\omega}{\nu}x}$ for $x < x_1^-$, then

$$\begin{aligned} u(x) &= \begin{cases} a_j + O(\delta^{1/2}) & x \in (x_j^-, x_j^+), \quad j = 1, 2, \dots, N, \\ a_j + b_j(x - x_j^+) + O(\delta^{1/2}) & x \in (x_j^+, x_{j+1}^-) \quad j = 0, 1, 2, \dots, N, \end{cases} \\ u'(x) &= \begin{cases} O(\delta) & x \in (x_j^-, x_j^+), \quad j = 1, 2, \dots, N, \\ b_j + O(\delta^{1/2}) & x \in (x_j^+, x_{j+1}^-) \quad j = 0, 1, 2, \dots, N, \end{cases} \end{aligned} \quad (4.7)$$

where $\mathbf{a} = (a_1, a_2, \dots, a_N)^\top$ is the corresponding eigenvector of the capacitance matrix C and $b_j = \frac{a_{j+1} - a_j}{s_j}$, $j = 1, 2, \dots, N - 1$. Here, $x_0^+ := -M$ and $x_{N+1}^- := M$ for a large enough M .

Proof. The proof proceeds in two steps. In Step 1, we derive the expressions for $u(x)$ and $u'(x)$ in (4.7) by mathematical induction. In Step 2, we determine the constants a_j and b_j .

Step 1. For $j = 0$, on interval (x_0^+, x_1^-) , $u(x) = e^{-i\frac{\omega}{v}x} = 1 + O(\delta^{1/2})$, $u'(x) = O(\delta^{1/2})$, so (4.7) holds for the case when $j = 0$. Now, suppose that for $x \in (x_j^+, x_{j+1}^-)$,

$$\begin{cases} u(x) = a_j + b_j(x - x_j^+) + O(\delta^{1/2}), \\ u'(x) = b_j + O(\delta^{1/2}). \end{cases}$$

It follows that

$$\begin{cases} u(x_{j+1}^-) = a_j + b_j(x_{j+1}^- - x_j^+) + O(\delta^{1/2}) := a_{j+1} + O(\delta^{1/2}), \\ \left. \frac{du}{dx} \right|_+(x_{j+1}^-) = \delta \left. \frac{du}{dx} \right|_-(x_{j+1}^-) = O(\delta). \end{cases}$$

Furthermore, from the propagation matrix, we have

$$\begin{aligned} u(x) &= \cos\left(kr(x - x_{j+1}^-)\right)u|_+(x_{j+1}^-) + \frac{1}{kr}\sin\left(kr(x - x_{j+1}^-)\right)u'|_+(x_{j+1}^-) \\ &= (1 + O(\delta))u|_+(x_{j+1}^-) + [(x - x_{j+1}^-) + O(\delta)]u'|_+(x_{j+1}^-) \\ &= a_{j+1} + O(\delta^{1/2}), \\ u'(x) &= -kr\sin\left(kr(x - x_{j+1}^-)\right)u|_+(x_{j+1}^-) + \cos\left(kr(x - x_{j+1}^-)\right)u'|_+(x_{j+1}^-) \\ &= O(\delta)u|_+(x_{j+1}^-) + (1 + O(\delta))u'|_+(x_{j+1}^-) \\ &= O(\delta) \end{aligned}$$

for $x \in (x_{j+1}^-, x_{j+1}^+)$. At the end point x_{j+1} , the jump relation gives

$$\begin{cases} u|_+(x_{j+1}^+) = a_{j+1} + O(\delta^{1/2}), \\ \left. \frac{du}{dx} \right|_+(x_{j+1}^+) = \frac{1}{\delta} \left. \frac{du}{dx} \right|_-(x_{j+1}^+) = O(1) = b_{j+1} + O(\delta^{1/2}). \end{cases}$$

Then, for $x \in (x_{j+1}^+, x_{j+2}^-)$, it holds that

$$\begin{aligned} u(x) &= \cos\left(k(x - x_{j+1}^+)\right)u|_+(x_{j+1}^+) + \frac{1}{k}\sin\left(k(x - x_{j+1}^+)\right)u'|_+(x_{j+1}^+) \\ &= (1 + O(\delta))u|_+(x_{j+1}^+) + [(x - x_{j+1}^+) + O(\delta)]u'|_+(x_{j+1}^+) \\ &= a_j + b_{j+1}(x - x_{j+1}^+) + O(\delta^{1/2}), \end{aligned}$$

and

$$\begin{aligned} u'(x) &= -kr\sin\left(kr(x - x_{j+1}^+)\right)u|_+(x_{j+1}^+) + \cos\left(kr(x - x_{j+1}^+)\right)u'|_+(x_{j+1}^+) \\ &= O(\delta)u|_+(x_{j+1}^+) + (1 + O(\delta))u'|_+(x_{j+1}^+) \\ &= b_{j+1} + O(\delta^{1/2}). \end{aligned}$$

This proves (4.7) by mathematical induction.

Step 2. Using the propagation matrix, we have

$$\begin{pmatrix} u(x_{j+1}^-) \\ u'(x_{j+1}^-) \end{pmatrix} = \begin{pmatrix} \cos(ks_j) & \frac{1}{k} \sin(ks_j) \\ -k \sin(ks_j) & \cos(ks_j) \end{pmatrix} \begin{pmatrix} \cos(krl_j) & \frac{\delta}{kr} \sin(krl_j) \\ -\frac{kr}{\delta} \sin(krl_j) & \cos(krl_j) \end{pmatrix} \begin{pmatrix} u(x_j^-) \\ u'(x_j^-) \end{pmatrix} \\ = \left[\begin{pmatrix} 1 - rl_j s_j \lambda & s_j \\ -rl_j \lambda & 1 \end{pmatrix} + O(\delta^{1/2}) \right] \begin{pmatrix} u(x_j^-) \\ u'(x_j^-) \end{pmatrix}, \quad j = 1, 2, \dots, N-1,$$

where $u'(x)$ is understood as $\frac{du}{dx}|_-$. By the formula (4.7), we also have

$$\begin{pmatrix} u(x_j^-) \\ u'(x_j^-) \end{pmatrix} = \begin{pmatrix} a_j \\ \frac{a_j - a_{j-1}}{s_{j-1}} \end{pmatrix} + O(\delta^{1/2}), \quad j = 2, \dots, N, \quad \begin{pmatrix} u(x_1^-) \\ u'(x_1^-) \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} + O(\delta^{1/2}).$$

Combining the two relations, we can obtain that

$$\frac{a_{j+1}}{rl_j s_j} + \frac{a_{j-1}}{rl_j s_{j-1}} = \left(\frac{1}{rl_j s_{j-1}} + \frac{1}{rl_j s_j} - \lambda \right) a_j, \quad j = 2, 3, \dots, N-1, \\ \frac{1}{rl_1 s_1} a_2 = \left(\frac{1}{rl_1 s_1} - \lambda \right) a_1.$$

If we imagine adding an interval of length s_N after the last resonator, we can use the propagation matrix to obtain

$$\begin{pmatrix} a_N \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - rl_N s_N \lambda & s_N \\ -rl_N \lambda & 1 \end{pmatrix} \begin{pmatrix} a_N \\ \frac{a_N - a_{N-1}}{s_{N-1}} \end{pmatrix}.$$

This yields

$$\frac{1}{rl_N s_{N-1}} a_{N-1} = \left(\frac{1}{rl_N s_{N-1}} - \lambda \right) a_N.$$

Combining these equations, we obtain $(C - \lambda I)\mathbf{a} = \mathbf{0}$, which completes the proof. $\square$

In the case $\delta \rightarrow \infty$, a similar characterization follows from Theorem 2.2.

Corollary 4.6. Under the assumption of Theorem 4.5, we choose $t = \delta_0^{1/2}$ in Theorem 2.2(3), such that for $x < x_1^-$, $v(x) = \delta_0^{1/2} u(x)$ is the corresponding eigenmode for $\delta = r^2/\delta_0$. Then, as $\delta_0 \rightarrow 0$ (namely, $\delta \rightarrow \infty$), we have

$$v(x) = \begin{cases} c_j + O(\delta^{-1/2}) & x \in (x_{j-1}^+, x_j^-) \quad j = 1, 2, \dots, N+1, \\ c_j + a_j(x - x_j^+) + O(\delta^{-1/2}) & x \in (x_j^-, x_j^+), \quad j = 1, 2, \dots, N. \end{cases} \quad (4.8)$$

Here, $\mathbf{a} = (a_1, a_2, \dots, a_N)^\top$ is the corresponding eigenvector of the capacitance matrix C , and $c_1 = 0$, $c_j = c_{j-1} + a_{j-1} \ell_{j-1}$ for $j = 2, \dots, N$.

4.3 | Comparison With the Three-Dimensional Case as $\delta \rightarrow \infty$

In this subsection, we present a concise comparison between the subwavelength resonant frequencies in the one- and three-dimensional settings. To this end, we consider a three-dimensional analog of the scattering problem (2.4); see [30].

In the three-dimensional case, assuming that k^2 is not a Dirichlet eigenvalue of $-\Delta$ on D , the scattering problem admits a resonant frequency if and only if the operator $\mathcal{A}(k, \delta)$ is not injective. Here, $\mathcal{A}(k, \delta) : L^2(\partial D) \times L^2(\partial D) \rightarrow H^1(\partial D) \times L^2(\partial D)$ is defined by

$$\mathcal{A}(k, \delta) = \begin{pmatrix} S_D^{rk} & -S_D^k \\ -\frac{1}{2}I + \mathcal{K}_D^{rk,*} & -\delta\left(\frac{1}{2}I + \mathcal{K}_D^{k,*}\right) \end{pmatrix},$$

where S_D^ω is the single layer potential and $\mathcal{K}_D^{\omega,*}$ represents the Neumann–Poincaré operator. Introducing the parameter $\tau = \delta^{-1}$, we observe that $\mathcal{A}(k, \delta)$ is injective if and only if $\tilde{\mathcal{A}}(k, \tau)$ is injective, where

$$\tilde{\mathcal{A}}(k, \tau) = \begin{pmatrix} S_D^{rk} & -S_D^k \\ -\tau\left(\frac{1}{2}I - \mathcal{K}_D^{rk,*}\right) & -\left(\frac{1}{2}I + \mathcal{K}_D^{k,*}\right) \end{pmatrix}.$$

Consequently, the subwavelength resonance problem in the limit $\delta \rightarrow \infty$ reduces to finding $\omega(\tau)$ such that $\omega(\tau) \rightarrow 0$ as $\tau \rightarrow 0$ and $\tilde{\mathcal{A}}(k, \tau)$ is not injective. It is straightforward to verify that $\tilde{\mathcal{A}}(k, \tau)$ is continuous in the norm topology. Examining the operator $\tilde{\mathcal{A}}(0, 0)$, we find that

$$\tilde{\mathcal{A}}(0, 0) = \begin{pmatrix} S_D^0 & -S_D^0 \\ 0 & -\left(\frac{1}{2}I + \mathcal{K}_D^{0,*}\right) \end{pmatrix}$$

is invertible, which follows from the invertibility of $S_D^0 : L^2(\partial D) \rightarrow H^1(\partial D)$ and the fact that the spectrum of $\mathcal{K}_D^{0,*}$ is contained in $(-1/2, 1/2]$; see again [30]. Since the set of invertible operators forms an open subset in the space of bounded operators equipped with the norm topology, $\tilde{\mathcal{A}}(k, \tau)$ remains invertible in a neighborhood of $(k, \tau) = (0, 0)$. This implies that the three-dimensional scattering problem possesses no subwavelength resonant frequencies in the limit $\delta \rightarrow \infty$. This behavior stands in stark contrast to the one-dimensional case, where Theorem 4.4 establishes the existence of exactly $2N$ subwavelength resonant frequencies as $\delta \rightarrow \infty$.

5 | Nonreciprocal System

In this section, we generalize the propagation matrix method to wave propagation in systems of non-Hermitian high-contrast resonators. We consider the same chain of resonators as in Section 2.1. By introducing an imaginary gauge potential γ (see [31, 32]), the governing equation for wave propagation becomes a generalized Sturm–Liouville equation: for $x \in \mathbb{R}$,

$$-\frac{\omega^2}{\kappa(x)}u(x) - \gamma(x)\frac{d}{dx}u(x) - \frac{d}{dx}\left(\frac{1}{\rho(x)}\frac{d}{dx}u(x)\right) = 0, \quad (5.1)$$

with $\gamma(x) = \gamma_i$ for $x \in D_i$, and 0 for $x \in \mathbb{R} \setminus D$. The other parameters are the same as in (2.2) and (2.3). In this setting, the wave problem (5.1) can be rewritten as the following system of coupled one-dimensional equations:

$$\begin{cases} \frac{d^2}{dx^2}u(x) + \gamma\frac{d}{dx}u(x) + \frac{\omega^2}{v_b^2}u(x) = 0, & x \in D_i, \\ \frac{d^2}{dx^2}u(x) + \frac{\omega^2}{v^2}u(x) = 0, & x \in \mathbb{R} \setminus D, \\ u|_+(x_i^\pm) = u|_-(x_i^\pm), & \text{for all } 1 \leq i \leq N, \\ \frac{du}{dx}\Big|_\pm(x_i^\mp) = \delta\frac{du}{dx}\Big|_\mp(x_i^\mp), & \text{for all } 1 \leq i \leq N, \\ \frac{du}{d|x|}(x) - i\frac{\omega}{v}u(x) = 0, & x \in (-\infty, x_1^-) \cup (x_N^+, \infty). \end{cases} \quad (5.2)$$

5.1 | Propagation Matrix

We now formulate the propagation matrix for the system (5.2). Consider the second-order ODE:

$$\frac{d^2}{dx^2}u(x) + \gamma \frac{d}{dx}u(x) + k^2u(x) = 0, \quad x \in (0, a), \quad (5.3)$$

which can be reformulated as the first-order ODE system $U'(x) = AU(x)$, where

$$A := \begin{pmatrix} 0 & 1 \\ -k^2 & -\gamma \end{pmatrix}, \quad U(x) := \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}.$$

Its solution is given by, for $x \in [0, a]$,

$$U(x) = e^{xA}U(0). \quad (5.4)$$

The matrix A has two eigenvalues given by

$$\lambda_1 = \frac{-\gamma - \sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-\gamma + \sqrt{\Delta}}{2}, \quad \text{where } \Delta = \gamma^2 - 4k^2.$$

It can be diagonalized as

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}, \quad P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

Therefore, we have the propagation matrix from $x = 0$ to $x = a$:

$$\begin{aligned} e^{aA} &= P \begin{pmatrix} e^{\lambda_1 a} & 0 \\ 0 & e^{\lambda_2 a} \end{pmatrix} P^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 a} - \lambda_1 e^{\lambda_2 a} & e^{\lambda_2 a} - e^{\lambda_1 a} \\ -\lambda_1 \lambda_2 (e^{\lambda_2 a} - e^{\lambda_1 a}) & \lambda_2 e^{\lambda_2 a} - \lambda_1 e^{\lambda_1 a} \end{pmatrix} \\ &= e^{-\frac{\gamma a}{2}} M(k) T(k; \gamma, a) M\left(\frac{1}{k}\right), \end{aligned}$$

where $M(\cdot)$ is given as in (2.7), and $T(\cdot)$ is defined by

$$T(k; \gamma, a) := \begin{pmatrix} \frac{\gamma}{\sqrt{\Delta}} \sinh\left(\frac{\sqrt{\Delta}a}{2}\right) + \cosh\left(\frac{\sqrt{\Delta}a}{2}\right) & \frac{2k}{\sqrt{\Delta}} \sinh\left(\frac{\sqrt{\Delta}a}{2}\right) \\ -\frac{2k}{\sqrt{\Delta}} \sinh\left(\frac{\sqrt{\Delta}a}{2}\right) & -\frac{\gamma}{\sqrt{\Delta}} \sinh\left(\frac{\sqrt{\Delta}a}{2}\right) + \cosh\left(\frac{\sqrt{\Delta}a}{2}\right) \end{pmatrix}.$$

Here, the factor $e^{-\frac{\gamma a}{2}}$ describes the attenuation of the wave during its propagation (see [33]). Furthermore, similar to the derivation of (2.14), we find that ω is a nontrivial resonant frequency of (5.2) if and only if $k = \omega/v$ satisfies

$$\begin{pmatrix} t \\ it \end{pmatrix} = \prod_{j=1}^N e^{-\frac{\gamma_j a}{2}} M\left(\frac{r}{\delta}\right) T(rk; \gamma_N, \ell_N) M\left(\frac{\delta}{r}\right) T(k; 0, s_{N-1}) M\left(\frac{r}{\delta}\right) \cdots T(rk; \gamma_1, \ell_1) M\left(\frac{\delta}{r}\right) \begin{pmatrix} s \\ -is \end{pmatrix} \quad (5.5)$$

for some $s, t \neq 0$. Note $T(kr; \gamma, l) = \frac{1}{r} T(k; \frac{\gamma}{r}, rl)$, and let

$$\mathbf{t} = (r\ell_1, s_1, r\ell_2, \dots, s_{N-1}, r\ell_N) \in \mathbb{R}_{>0}^{2N-1}, \quad \boldsymbol{\beta} = \left(\frac{\gamma_1}{r}, 0, \frac{\gamma_2}{r}, \dots, 0, \frac{\gamma_N}{r}\right), \quad \sigma = \frac{\delta}{r}.$$

Then, equation (5.5) reduces to finding all $k \in \mathbb{C}$ satisfying

$$t \begin{pmatrix} 1 \\ i \end{pmatrix} = M\left(\frac{1}{\sigma}\right) T(k; \beta_{2N-1}, t_{2N-1}) M(\sigma) T(k; \beta_{2N-2}, t_{2N-2}) M\left(\frac{1}{\sigma}\right) \cdots M\left(\frac{1}{\sigma}\right) T(k; \beta_1, t_1) M(\sigma) \begin{pmatrix} 1 \\ -i \end{pmatrix}. \quad (5.6)$$

Let $e_+ = (1, i)^T$ and $e_- = (1, -i)^T$. We observe that

$$T(k; \beta_i, t_i) e_{\pm} = \left(\pm \frac{2ki}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) + \cosh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) \right) e_{\pm} + \frac{\beta_i}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) e_{\mp};$$

$$M(\sigma) e_{\pm} = \frac{1+\sigma}{2} e_{\pm} + \frac{1-\sigma}{2} e_{\mp}.$$

Therefore, under the basis $\{e_+, e_-\}$, (5.6) can be rewritten as

$$t \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R\left(\frac{1}{\sigma}\right) L_{2N-1}(k) R(\sigma) L_{2N-2}(k) R\left(\frac{1}{\sigma}\right) \cdots R\left(\frac{1}{\sigma}\right) L_1(k) R(\sigma) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad t \neq 0, \quad (5.7)$$

where $R(z)$ is defined as in (2.18), and

$$L_i(k) := \begin{pmatrix} \frac{2ki}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) + \cosh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) & \frac{\beta_i}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) \\ \frac{\beta_i}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) & -\frac{2ki}{\sqrt{\Delta_i}} \sinh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) + \cosh\left(\frac{\sqrt{\Delta_i} t_i}{2}\right) \end{pmatrix}.$$

As in the previous sections, Equation (5.7) (or its normalized version) forms the basis for deriving the resonances of problem (5.2). However, unlike the Hermitian case, the matrix L_i is not diagonal, which makes the characteristic function for resonant frequencies more intricate than $f(k; \sigma)$ in Theorem 3.3. Consequently, we focus on subwavelength resonances and leave the analysis of the non-subwavelength regime for future work.

5.2 | Capacitance Matrix Theory

In this section, we develop the capacitance matrix theory for the subwavelength resonances of problem (5.2), building on the propagation matrix framework.

Similarly to (3.1), define the modified total propagation matrix associated with (5.2):

$$M_{tot}(k; \sigma) := \frac{(4\sigma)^N}{(1+\sigma)^{2N}} R\left(\frac{1}{\sigma}\right) L_{2N-1}(k) R(\sigma) L_{2N-2}(k) \cdots R\left(\frac{1}{\sigma}\right) L_1(k) R(\sigma),$$

where, using (3.8),

$$\frac{4\sigma}{(1+\sigma)^2} P R\left(\frac{1}{\sigma}\right) L(k) R(\sigma) P = (R + \nu S) L(k) (R + \nu S), \quad \text{with } P = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

Thus, it follows that

$$P M_{tot}(k; \sigma) P = G(k; \nu),$$

with

$$G(k, \nu) := (R + \nu S) L_{2N-1}(k) (R + \nu S) L_{2N-2}(k) (R + \nu S) \cdots L_1(k) (R + \nu S). \quad (5.8)$$

By (5.7), $\omega = kv$ is a resonant frequency if and only if k is a zero of

$$g(k, \nu) := G(k, \nu)_{2,2} = g_0(k) + g_1(k)\nu + g_2(k)\nu^2 + \cdots + g_{2N}(k)\nu^{2N}.$$

Similarly to what we have done in Section 4, applying the Newton polygon method to the above expansion of $g(k, \nu)$, we can recover the capacitance matrix theory for the subwavelength resonances in [31]. We define the generalized capacitance matrix as

$$C := \begin{pmatrix} \theta_1 & -\theta_1 & & & & \\ -\theta_2 & \theta_2 + \theta_3 & -\theta_3 & & & \\ & -\theta_4 & \theta_4 + \theta_5 & -\theta_5 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\theta_{2N-4} & \theta_{2N-4} + \theta_{2N-3} & -\theta_{2N-3} \\ & & & & -\theta_{2N-2} & \theta_{2N-2} \end{pmatrix}, \quad (5.9)$$

where

$$\theta_j := \begin{cases} \frac{\beta_j e^{\frac{\beta_j t_j}{2}}}{2 \sinh\left(\frac{\beta_j t_j}{2}\right) t_{j+1}}, & j = 2i - 1, \quad i = 1, 2, \dots, N, \\ \frac{\beta_{j+1} e^{-\frac{\beta_{j+1} t_{j+1}}{2}}}{2 \sinh\left(\frac{\beta_{j+1} t_{j+1}}{2}\right) t_j}, & j = 2i, \quad i = 0, 1, 2, \dots, N - 1, \end{cases} \quad (5.10)$$

where $t_0 = t_{2N} = 1$.

Remark. Noting that $\mathbf{t} = (r\ell_1, s_1, r\ell_2, \dots, s_{N-1}, r\ell_N)$, and $\boldsymbol{\beta}$ is given by $\boldsymbol{\beta} = (\frac{\gamma_1}{r}, 0, \frac{\gamma_2}{r}, \dots, 0, \frac{\gamma_N}{r})$. Substituting these into (5.9) and (5.10), a direct computation yields the explicit form of C , which is related to the gauge capacitance matrix C^γ in [31, Corollary 2.6] by $C = V^{-1}C^\gamma/r$.

Theorem 5.1. Let $\delta \in \mathbb{C}$. For cases when $\delta \rightarrow 0$, the scattering problem (5.2) has exactly $2N$ subwavelength resonant frequencies:

- a trivial frequency $\omega^* = 0$.
- the first eigenfrequency $\omega_1(\delta)$, which is an analytic function of δ (or δ^{-1}) with leading asymptotic expansion:

$$\omega_1(\delta) = O(\delta^2) + \frac{-i\delta v_b^2 \left[\prod_{j=1}^N \left(\frac{\gamma_{2j-1}}{2} e^{\frac{\gamma_{2j-1} r \ell_{2j-1}}{2}} \right) + \prod_{j=1}^N \left(\frac{\gamma_{2j-1}}{2} e^{-\frac{\gamma_{2j-1} r \ell_{2j-1}}{2}} \right) \right]}{v \sum_{i=1}^N \left[\sinh\left(\frac{\gamma_{2i-1} r \ell_{2j-1}}{2}\right) \prod_{j=1}^{i-1} \left(\gamma_{2j-1} e^{-\frac{\gamma_{2j-1} r \ell_{2j-1}}{2}} \right) \prod_{j=i+1}^N \left(\gamma_{2j-1} e^{\frac{\gamma_{2j-1} r \ell_{2j-1}}{2}} \right) \right]}.$$

- the remaining $2N - 2$ frequencies are analytic functions of $\delta^{1/2}$, with leading-order asymptotic expansions:

$$\omega_i^\pm(\delta) = \pm v \sqrt{\frac{\lambda_i}{r}} \delta - i\delta \frac{v_b}{2} \frac{\theta_0 a_{i1} b_{i1} + \theta_{2N-1} a_{iN} b_{iN}}{\sum_{j=1}^N a_{ij} b_{ij}} + O(\delta^{\frac{3}{2}}), \quad i = 2, 3, \dots, N,$$

where $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N$ are the eigenvalues of C , and $\mathbf{a}_i = (a_{ij})_{j=1}^N^\top$, $\mathbf{b}_i = (b_{ij})_{j=1}^N$ are the right and left eigenvectors corresponding to λ_i , respectively.

What's more, the eigenmodes corresponding to the resonant frequencies $\omega_i = k_i v$ with $k_i = \sqrt{\lambda_i \delta} + O(\delta)$, $i = 1, \dots, N$, have the same form as (4.7), where $\mathbf{a} = (a_1, a_2, \dots, a_N)^\top$ is the corresponding eigenvector of the capacitance matrix C and $b_j = \frac{a_{j+1} - a_j}{s_j}$ for $j = 1, 2, \dots, N - 1$.

6 | Concluding Remarks

In this work, we have analyzed the scattering resonances of general one-dimensional acoustic media, identifying them as the zeros of an explicit trigonometric polynomial using the propagation matrix method. Leveraging Nevanlinna theory, we have established the global distribution properties of these resonances and characterized the resonance-free region by demonstrating the uniform boundedness of their imaginary parts. In addition, we have derived asymptotic expansions for both subwavelength and non-subwavelength resonances in terms of the high contrast parameter. In the subwavelength regime, we have further employed the Newton polygon method to establish connections between our approach and the capacitance matrix theory for Minnaert resonances. This work lays the foundation for the analysis of scattering resonances in acoustic media with high-contrast bulk modulus, as well as the extension of capacitance matrix theory to non-subwavelength resonances. These topics will be investigated in forthcoming papers.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data and codes supporting the findings of this work are available upon request.

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Appendix A: Asymptotic Analysis of Zeros via Newton Polygon Method

In this section, we review the Newton polygon method [34] for obtaining the asymptotics of the zeros of an analytic function. We consider a function $f(u, v)$ that is analytic in a region $U \subset \mathbb{C}^2$, where u is treated as the variable and v as a parameter. Suppose that for $v = v_0$, the function $f(\cdot, v_0)$ has an n -fold zero at $u = u_0$:

$$\frac{\partial^k f}{\partial u^k}(u_0, v_0) = 0, \quad \text{for } k = 0, \dots, n-1, \quad \frac{\partial^n f}{\partial u^n}(u_0, v_0) \neq 0.$$

We next analyze the asymptotic behavior of the zeros $u(v)$ of $f(\cdot, v)$ as $v \rightarrow v_0$ using the Newton polygon method. For simplicity, we set $(u_0, v_0) = (0, 0)$. Expanding f at $(0, 0)$, we have

$$f(u, v) = \sum_{j,k \geq 0} c_{jk} u^j v^k, \quad c_{j0} = 0 \text{ for } j < n, \text{ and } c_{n0} \neq 0. \quad (\text{A.1})$$

It suffices to find the asymptotic behavior of solutions $u(v)$ to $f(u(v), v) = 0$ as $v \rightarrow 0$. The Newton polygon is constructed as follows:

1. For each nonzero coefficient c_{jk} in (A.1), plot (j, k) in the plane $\mathbb{R}^2$.
2. Form the convex hull of the set $\{(j, k) : c_{jk} \neq 0\}$.
3. The lower boundary of this convex hull consists of a sequence of linear segments (edges), which determines the dominant asymptotic regimes.

Each edge l connecting points (j_1, k_1) and (j_2, k_2) with $j_1 < j_2$ has the slope

$$\lambda_l = \frac{k_2 - k_1}{j_2 - j_1}. \quad (\text{A.2})$$

The dominant scaling exponent for v corresponding to this edge is $s_l = -\lambda_l$, implying the asymptotic form $u \sim cv^{s_l}$ with $c \neq 0$. The dominant balance equation for edge l is derived as follows:

1. Selecting all terms (j, k) lying on the line l .
2. Substituting the asymptotic form $u = cv^{s_l}$ in the equation

$$\sum_{(j,k) \in l \cap \mathbb{Z}^2} c_{jk} u^j v^k = 0$$

to obtain the balance equation in c

$$\sum_{(j,k) \in l \cap \mathbb{Z}^2} c_{jk} c^j = 0. \quad (\text{A.3})$$

3. The left side of (A.3) is a polynomial in c . The number of nonzero solutions c (counting multiplicities) equals to the number of asymptotic branches associated with l .

For each asymptotic branch, the full expansion is given by a Puiseux series

$$u(v) = \sum_{m=0}^{\infty} d_m v^{s_l + t_m}, \quad (\text{A.4})$$

where $0 = t_0 < t_1 < t_2 < \dots$ are rational exponents determined recursively. The coefficients d_m are determined as follows:

1. Substitute the series (A.4) into $f(u, v) = 0$ and expand it in powers of v .
2. Match terms at each order of v and solve sequentially for each d_m .

The exponents t_m are determined by the geometry of the Newton polygon above the initial edge. By Rouché's Theorem, the total number of asymptotic branches (counting multiplicities) must equal n , consistent with the multiplicity of the zero u_0 of $f(\cdot, v_0)$.

Appendix B: Proof of Proposition 4.2

Proof. Since $g(k; \nu) := G(k, \nu)_{2,2}$, it suffices to derive the following expansions of $G_l(k)$ in (3.11) as $k \rightarrow 0$: for $1 \leq l \leq N-1$,

$$\begin{aligned} G_l(k) = & \left(\prod_{j=1}^{2N-1} t_j \right) \left[2^l (-2ik)^{2N-1-2l} \sum_{1 \leq j_1 < j_2 < \dots < j_l \leq 2N-2} \left(\prod_{i=1}^l \theta_{j_i} \right) R + \theta_0 2^{l-1} (-2ik)^{2N-2l} \sum_{2 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-2} \left(\prod_{i=1}^{l-1} \theta_{j_i} \right) R_- \right. \\ & \left. + \theta_{2N-1} 2^{l-1} (-2ik)^{2N-2l} \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N-3} \left(\prod_{i=1}^{l-1} \theta_{j_i} \right) R_+ + O(k^{2N-2l+1}) \right], \end{aligned} \quad (\text{B.1})$$

and for $l = 0, N$,

$$G_0(k) = \left(\prod_{j=1}^{2N-1} t_j \right) (-2ki)^{2N-1} R + O(k^{2N+1}), \quad G_N(k) = 2^{N-1} (R_+ + R_-) + O(k). \quad (\text{B.2})$$

$$\begin{array}{ccccccc} \boxed{SL_{2N-1}} & R & \boxed{L_{2N-2}} & R & \boxed{L_{2N-3}SL_{2N-4}SL_{2N-5}} & \cdots & R & \boxed{L_4SL_3} & R & \boxed{L_2SL_1S} \\ M_L^1 & & M_1^0 & & M_2^2 & & & M_{2N-1-l}^1 & & M_R^2 \end{array}$$

FIGURE B.1 | Structure of Q with $a_L = 1, a_1 = 0, a_2 = 2, \dots, a_{2N-l-1} = 1, a_R = 2$.

This expansion is based on (3.9) and the following identities:

$$\begin{aligned} 1. & RMR = \eta(M)R, \\ 2. & \eta(L_j) = -2t_j k i + O(k^3), \quad \eta(L_{j+1}SL_j) = 2 + O(k^2), \\ 3. & \tau_0(\eta(L_j)) = 1, \quad \tau_0(\eta(L_{j+1}SL_j)) = 0, \quad \tau_0(\eta(L_{j+2}SL_{j+1}SL_j)) \geq 1, \\ 4. & SL_{2N-1}(k)R = R_- + O(k), \quad RL_1(k)S = R_+ + O(k), \end{aligned} \quad (\text{B.3})$$

where $\eta(M) = M_{21} + M_{22} - M_{11} - M_{12}$ for $M \in \mathbb{C}^{2 \times 2}$, and $\tau_0(f)$ denotes the order of the leading-order term in the asymptotic expansion of $f(k)$ as $k \rightarrow 0$, in particular, $\tau_0(0) = +\infty$.

First, for $1 \leq l \leq N-1$, by (3.9), the expression of $G_l(k)$ consists of $\binom{2N}{l}$ terms, each corresponding to a selection of l S -matrices (or equivalently, $2N-l$ R -matrices) from the $2N$ available matrices $R + \nu S$. We denote by Q the matrix product corresponding to a specific selection, and we will analyze its order in k as $k \rightarrow 0$.

To this end, we first consider the $2N-l-1$ gaps formed by the $2N-l$ R -matrices and denote the product of all matrices within each gap by $M_1^{a_1}, M_2^{a_2}, \dots, M_{2N-l-1}^{a_{2N-l-1}}$, where a_i denotes the number of S -matrices in the i th gap (see Figure B.1 for an illustration). Moreover, we denote by $M_L^{a_L}$ the product of all matrices to the left of the first R -matrix, and by $M_R^{a_R}$ the product of all matrices to the right of the last R -matrix, where a_L and a_R denote the corresponding numbers of S -matrices involved. In particular, M_L^0 and M_R^0 are identity matrices. Then, by (B.3)(1), the total matrix product Q can be written as

$$Q = \left[\prod_{i=1}^{2N-l-1} \eta(M_i^{a_i}) \right] M_L^{a_L} R M_R^{a_R}. \quad (\text{B.4})$$

Since we select exactly l S -matrices, we have $a_L + a_R + \sum_{i=1}^{2N-l-1} a_i = l$, which implies that at least $2N-2l-1$ of the a_i 's must be zero. Furthermore, applying (B.3)(3) to (B.4) and using the definition of τ_0 , we obtain

$$\tau_0(Q) \geq \sum_{i=1}^{2N-l-1} \tau_0(\eta(M_i^{a_i})) \geq \sum_{i=1}^{2N-l-1} \tau_0(\eta(M_i^{a_i})) \mathbf{1}_{a_i=0} \geq 2N-2l-1. \quad (\text{B.5})$$

The equality holds if and only if we have $2N-2l-1$ a_i 's equal to zero, the remaining l a_i 's are one, and $a_L = a_R = 0$. In this case (i.e., $\tau_0(Q) = 2N-2l-1$), the l matrices $M_i^{a_i}$ with $a_i = 1$ have the form $L_{j_i+1}SL_{j_i}$ with $1 \leq j_1 < j_2 < \dots < j_l \leq 2N-2$. Then, according to (B.3)(2), the leading-order term in the expansion of Q is given by

$$\begin{aligned} Q &= \left[\prod_{i=1}^{2N-l-1} \eta(M_i^{a_i}) \right] R = \prod_{j \neq j_i, j_{i+1}} \eta(L_j) \prod_{i=1}^l \eta(L_{j_i+1}SL_{j_i}) R \\ &= \left[2^l (-2ik)^{2N-1-2l} \prod_{j=1}^{2N-1} t_j \prod_{i=1}^l \theta_{j_i} \right] R + O(k^{2N-2l+1}). \end{aligned} \quad (\text{B.6})$$

Now consider the selections that yield $\tau_0(Q) = 2N-2l$. In this case, the equality condition in (B.5) implies that exactly $2N-2l$ of the a_i 's equal to zero. Since $a_L + a_R + \sum_{i=1}^{2N-l-1} a_i \mathbf{1}_{a_i \neq 0} = l$, it suffices to consider the following two cases:

- $a_L = a_R = 0$, exactly one a_i equals to two, and all remaining nonzero a_i 's equal to one. Then,

$$\tau_0(Q) = \sum_{i=1}^{2N-l-1} \tau_0(\eta(M_i^{a_i})) = \sum_{i=1}^{2N-l-1} \tau_0(\eta(M_i^{a_i})) (\mathbf{1}_{a_i=0} + \mathbf{1}_{a_i=2}) \geq 2N-2l+1.$$

In this case, $\tau_0(Q)$ cannot be $2N-2l$.

• $a_L = 1$, $a_R = 0$ or $a_R = 1$, $a_L = 0$, and all nonzero a_i equal to one. We first expand Q for the case $a_L = 1$, $a_R = 0$. In this case, the $l - 1$ matrices $M_i^{a_i}$ with $a_i = 1$ have the form $L_{j_i+1}SL_{j_i}$, where $1 \leq j_1 < j_2 < \dots < j_{l-1} \leq 2N - 3$. Then, using (B.3)(2)(4), we obtain

$$Q = 2^{l-1}(-2ik)^{2N-2l}\theta_{2N-1} \prod_{j=1}^{2N-1} t_j \prod_{i=1}^{l-1} \theta_{j_i} R_+ + O(k^{2N-2l+1}). \quad (\text{B.7})$$

Similarly, we can expand Q for the case when $a_R = 1$, $a_L = 0$ that

$$Q = 2^{l-1}(-2ik)^{2N-2l}\theta_0 \prod_{j=1}^{2N-1} t_j \prod_{i=1}^{l-1} \theta_{j_i} R_- + O(k^{2N-2l+1}).$$

For the remaining selections, we have $\tau_0(Q) \geq 2N - 2l + 1$, and they are absorbed into the $O(k^{2N-2l+1})$ term in (B.1). Having established the expansion of $G_l(k)$ for $1 \leq l \leq N - 1$, the proof is completed by deriving the expressions at the boundary indices $l = 0$ and $l = N$.

For $l = 0$, there is only one possible selection (i.e., selecting all R -matrices from the $2N$ available matrices $R + \nu S$), which gives

$$G_0(k) = \left[\prod_{i=1}^{2N-l-1} \eta(L_i) \right] R = \left(\prod_{j=1}^{2N-1} t_j \right) (-2ki)^{2N-1} R + O(k^{2N+1}).$$

For $l = N$, for any selection and the corresponding product Q , it follows from the inequality (B.5) that when $\tau_0(Q) = 0$, all a_i 's are equal to one, and either a_L or a_R is equal to one. By absorbing the product of other selections into the $O(k)$ term, we obtain $G_N(k) = 2^{N-1}(R_+ + R_-) + O(k)$ as desired. $\square$