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Inverse random scattering for the one-dimensional Helmholtz equation

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Abstract

This paper concerns the inverse random potential scattering problem for the Helmholtz equation in one dimension. The random potential is assumed to be a generalized microlocally isotropic Gaussian random field, whose covariance is a classical pseudodifferential operator. To investigate the inverse problem, as an effective mathematical tool, the scattering theory is employed to obtain an analytic domain and estimates for the resolvent of the elliptic operator with rough potentials. For the inverse problem, based on the resolvent estimates, we show that the principal symbol of the covariance operator can be reconstructed by a single realization of the boundary data averaged over the high-frequency band almost surely. Consequently, by analyticity of the resolvent, the uniqueness of the inverse problem can be achieved by data at multiple frequencies only in a finite interval. The method developed here is unified and can be applied to other stochastic inverse scattering problems in higher dimensions for various wave equations by boundary measurements.

Keywords: Helmholtz equation, Gaussian random field, reconstruction formula, boundary measurement, uniqueness

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1. Introduction

The inverse scattering in one dimension arise in many scientific and industrial areas such as medical imaging, geophysical exploration, non-destructive industrial testing and quantum physics. The aim is to determine the unknown scatterer from appropriate measurements. Driven by the significant applications, the inverse scattering problems on the line have been extensively investigated by many researchers. Consequently, a great deal of mathematical and numerical results are available, especially for the acoustic waves or Helmholtz equation [5, 7, 9, 11, 20].

In environments involving uncertainties, it is desirable to describe the scatterer as a random field. As a result, the scattered wave field is also random. Compared with deterministic counterparts, there are two major differences for stochastic scattering problems. First, the scatterer may be too rough to exist pointwisely and should be understood in the sense of distribution. Second, it is less meaningful to characterize the scatterer by a particular realization. Thus, the recovery of the statistics such as the mean and variance of the random scatterer is more interesting in stochastic inverse scattering, which are used to quantify the uncertainties of the scatterer. Recently, the inverse random scattering problems have received increasing attention [6, 12, 13, 16]. In particular, there are some results on the linear inverse random source problems in one dimension [3, 4, 15, 17]. However, to our knowledge, there have been no results on the inverse random potential scattering problems in one dimension. Such problems are highly nonlinear and ill-posed.

In this paper, we consider the following scattering problem characterized by the one-dimensional Helmholtz equation

$$\begin{cases} u'' + k^2 u + qu = 0, \quad \text{in} \quad \mathbb{R}, \\ u = \mathrm{e}^{\mathrm{i}kx} + u^{sc}, \end{cases} \tag{1.1}$$

where u^{sc} is the scattered field, and q is the random potential function supported in $(-1, 1)$. As a consequence, u^{sc} is also random. In the one-dimensional case, the outgoing wave radiation condition is equivalent to the following boundary conditions:

$$u^{sc} = \begin{cases} a^+ \mathrm{e}^{\mathrm{i}kx}, & x \geqslant 1, \\ a^- \mathrm{e}^{-\mathrm{i}kx}, & x \leqslant -1, \end{cases} \tag{1.2}$$

where $a^+, a^- \in \mathbb{R}$ are two random constants.

In this work, the potential is assumed to be a generalized microlocally isotropic Gaussian (GMIG) random field, whose covariance is described by a classical pseudodifferential operator of order $-m$ with the principal symbol given by $\mu(x)|\xi|^{-m}$. Here $\mu(x)$ is a smooth function supported in a bounded interval. It is said to be the micro-correlation strength of the random potential. Such random fields contain stochastic process such as the fractional Brownian motion and Markov field. Uniqueness for such random potentials was obtained in higher dimensions [6, 12]. An inverse random source problem in one dimension was discussed in [15]. However, the uniqueness was only obtained for Helmholtz equation with constant damping. In fact, [15] utilized the method developed in [12] to derive an integral identity connecting the potential or source with the high-frequency limit of expectation of the data, which however did not yield uniqueness for the case without damping in one dimension. Therefore, sometimes it is not necessary that the scattering on the line is easier than in higher dimensions.

The investigation of the inverse scattering problems relies on the direct scattering problems. Specifically, appropriate upper bounds for the resolvent of the elliptic operator with respect to

the wavenumber is required [6, 12, 14]. A commonly used method to derive resolvent estimates between weighted Sobolev spaces is the limiting absorption principle [1, 19]. However, the resolvent investigated by this method cannot be meromorphically extended to the lower-half complex plane. In this work, we employ scattering theory in [10] to derive useful resolvent estimates and an analytic domain for the resolvent. It has been realized that, the use of the scattering theory is an effective method to investigate the inverse scattering problems. Indeed, by considering the resolvent as a family of $L^2_{\text{comp}} \to L^2_{\text{loc}}$ operators, the meromorphic continuation of the resolvent becomes possible. On the other hand, in inverse scattering, the unknown scatterer usually has a compact support and the measurements are required to be taken on compact sets due to practical applications. Hence, to investigate inverse scattering problems, the scattering theory seems to be more convenient and effective to apply. In particular, the complex analytic domain of the resolvent allows the analytic continuation of the data from low frequencies to high frequencies, which is crucial for the derivation of uniqueness and stability in inverse scattering by using multi-frequency data only in a finite band [5, 18].

In this paper, we propose a new method to solve the inverse random potential scattering problem in one dimension. The strategy is to multiply the Helmholtz equation by plane waves and integrate by parts, which yields an integral identity connecting the Fourier transform of the random potential and the scattered field at a single boundary point. Then by using the ergodicity, the Fourier transform of the strength in the principal symbol of the random potential can be explicitly reconstructed by a single realization of the correlation boundary data averaged over the high-frequency band almost surely, which implies the uniqueness. It should be pointed out that our method only needs data at a single point, while results in [12, 15] require the data to be measured in bounded domains. Moreover, by analyticity of the resolvent, we obtain the uniqueness of the inverse problem by using data given only in a finite interval of frequencies, while uniqueness results in [12, 15] require data at all high frequencies. As an application of our new method, an open problem proposed in [15] on the inverse random source problem without damping can be solved. This method can also be applied to stochastic inverse problems in higher dimensions. In summary, the techniques developed in this work differ significantly from those in the existing works such as [6, 12] in the following two aspects. First, we employ scattering theory but not limiting absorption principle to investigate the resolvent of the elliptic operator. As a consequence, we obtain useful resolvent estimates and an analytic domain for the resolvent. Second, we develop a novel method to study the inverse problem by using only boundary measurements.

This paper is organized as follows. In section 2, some preliminaries are given for the GMIG random fields. In section 3, the meromorphic continuation of the resolvent is investigated. Section 4 is devoted to the inverse random potential scattering problem. An inverse random source problem is discussed in section 5.

2. Preliminaries

In this section, we first introduce some notations used in this paper. Let $W^{s,p} := W^{s,p}(\mathbb{R})$ stand for standard Sobolev spaces and let $H^s := W^{s,2}$. The function spaces $H^s_{\text{loc}} := H^s_{\text{loc}}(\mathbb{R})$ and $H^s_{\text{comp}} := H^s_{\text{comp}}(\mathbb{R})$ are defined as follows

$$H^s_{\text{loc}}(\mathbb{R}) := \{u \in \mathcal{D}'(\mathbb{R}) : \chi u \in H^s(\mathbb{R}), \forall \chi \in C_0^\infty(\mathbb{R})\},$$
$$H^s_{\text{comp}}(\mathbb{R}) := \{u \in H^s(\mathbb{R}) : \exists \chi \in C_0^\infty(\mathbb{R}), \chi u = u\}.$$

In this paper, we use $a \lesssim b$ to denote $a \leqslant Cb$, where $C > 0$ is a generic constant whose specific value may change step by step. Moreover, the symbol $A \subset\subset B$ means the closure of the set A is compact which is contained in B.

We next introduce the generalized Gaussian random fields. Let $\mathcal{D} := C_0^\infty(\mathbb{R})$ be the space of test functions with locally convex topology, whose dual space is denoted by $\mathcal{D}'$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. A random field $g : \Omega \to \mathcal{D}'$ is said to be a generalized Gaussian random field if $\omega \mapsto \langle g(\omega), \phi\rangle$ is a Gaussian random variable for all $\phi \in \mathcal{D}$. The expectation $\mathbb{E}g \in \mathcal{D}'$ and covariance $\mathrm{Cov}\, g : \mathcal{D} \times \mathcal{D} :\to \mathbb{R}$ are respectively defined by

$$\begin{aligned} &\mathbb{E}g : \phi \mapsto \mathbb{E}\langle g, \phi\rangle, \quad \phi \in \mathcal{D}, \\ &\mathrm{Cov}\, g : (\phi_1, \phi_2) \mapsto \mathrm{Cov}\left(\langle g, \phi_1\rangle, \langle g, \phi_2\rangle\right), \quad \phi_1, \phi_2 \in \mathcal{D}. \end{aligned}$$

The covariance operator C_g is defined by

$$\langle C_g\phi_1, \phi_2\rangle = \mathrm{Cov}\left(\langle g, \phi_1\rangle, \langle g, \phi_2\rangle\right), \quad \phi_1, \phi_2 \in \mathcal{D}.$$

Let $K_g(x,y)$ be the Schwarz kernel of C_g, i.e.

$$\langle C_g\phi_1, \phi_2\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi_2(y) K_g(x,y) \phi_1(x)\, \mathrm{d}x\mathrm{d}y.$$

Let I be an open interval. Now we state the definition of GMIG random fields.

Definition 2.1. A generalized Gaussian random field f with zero expectation is said to be microlocally isotropic of order $-m \in \mathbb{R}$ in $I \subset \mathbb{R}$ if its covariance operator C_g is a classical pseudodifferential operator of order $-m$ with the principal symbol $\mu(x)|\xi|^{-m}$, where $\mu(x) \in C_0^\infty(\mathbb{R})$ is a real-valued function with $\mathrm{supp}\,\mu \subset\subset I$. Here μ is called the micro-correlation strength of f.

Let c_g be the symbol of the operator C_g, i.e.

$$C_g(\phi)(x) = (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\cdot\xi} c_g(x,\xi)\, \hat{\phi}(\xi)\, \mathrm{d}\xi, \quad \phi \in \mathcal{D},$$

which gives

$$K_g(x,y) = (2\pi)^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} c_g(x,\xi)\, \mathrm{d}\xi.$$

The following lemma proved in [13] concerns the regularity of the microlocally isotropic Gaussian random field.

Lemma 2.1. *Let g be a microlocally isotropic Gaussian random field of order $-m$ in an open interval $I \subset \mathbb{R}$. Then $g \in W^{\frac{m-1}{2}-\delta,p}(I)$ almost surely for all $\delta > 0$ and $1 < p < \infty$.*

We obtain the following representation of the kernel by [6].

Lemma 2.2. *Let g be a GMIG random field of order $-m$ in I. The kernel $K_g(x,y)$ has the following form:*

(i) *If $a < m \leqslant a+1$ with $a = 1, 2, 3, \ldots$,*

$$K_g(x,y) = c\mu(x)|x-y|^{m-1} + F_m(x,y),$$

where c is a constant dependent on m and $F_m(x,y) \in C^{a,\gamma}(\mathbb{R}\times\mathbb{R};\mathbb{R})$ with $\gamma \in (0, m-a)$.

(ii) *If $0 < m < 1$,*

$$K_g(x,y) = c\mu(x)|x-y|^{m-1} + F_m(x,y),$$

where c is a constant dependent on m and $F_m(x,y) \in C^{0,\gamma}(\mathbb{R}\times\mathbb{R};\mathbb{R})$ with $\gamma \in (0, m)$.

(iii) *If $m = 1$,*

$$K_g(x,y) = c\mu(x)\log|x-y| + F_m(x,y),$$

where c is a constant dependent on $F_m(x,y) \in C^{0,\gamma}(\mathbb{R}\times\mathbb{R};\mathbb{R})$ with $\gamma \in (0,1)$.

3. Meromorphic continuation and estimates for the resolvent

In this section, we investigate the meromorphic continuation of the resolvent by scattering theory assuming that q is a GMIG random field of order $-m$ in $(-1,1)$ with $m>0$.

We begin with the free resolvent. The Green function of the Helmholtz equation is given by

$$G(x,y,k) = \frac{i}{2k}\mathrm{e}^{\mathrm{i}k|x-y|}.$$

Then denote the free resolvent operator by $\mathcal{R}_0(k) = (-D_x^2 - \lambda^2)^{-1}$ which has an explicit representation as follows

$$\mathcal{R}_0(k)(f) := \int_{\mathbb{R}} G(x,y,k) f(y)\,\mathrm{d}y.$$

The following lemma for the free resolvent is important in the subsequent analysis.

Lemma 3.1. *The resolvent $\mathcal{R}_0(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Im\lambda > 0$ as a family of operators*

$$\mathcal{R}_0(\lambda) : H^s(\mathbb{R}) \to H^t(\mathbb{R})$$

for all $s \in \mathbb{R}$ and $t \in [s, s+2]$. Furthermore, given a cut-off function $\chi \in C_0^\infty(\mathbb{R})$, $\chi\mathcal{R}_0(\lambda)\chi$ can be extended to a family of analytic operators for $\lambda \in \mathbb{C}\backslash\{0\}$ as follows

$$\chi\mathcal{R}_0(\lambda)\chi : H^s(\mathbb{R}) \to H^t(\mathbb{R})$$

with the resolvent estimates

$$\|\chi\mathcal{R}_0(\lambda)\chi\|_{\mathcal{L}(H^s,H^t)} \lesssim |\lambda|^{-1}(1+|\lambda|)^{t-s}\mathrm{e}^{L(\Im\lambda)_-},$$

where $t \in [s, s+2]$, $v_- := \max\{-v, 0\}$, L is a constant larger than the diameter of the support of χ and the norm $\|\cdot\|_{\mathcal{L}(H^s,H^t)}$ is the operator norm from H^s to H^t.

Proof. We begin with proving the resolvent estimates. It is straightforward to have

$$|G(x,y,\lambda)| \leqslant \frac{1}{2|\lambda|}\mathrm{e}^{-\Im\lambda|x-y|} \tag{3.1}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$. In order to obtain the resolvent estimates, we first consider the case $s = t = 0$. By using the estimate (3.1) and Schur's lemma [10, appendix A.5], we have

$$\|\chi\mathcal{R}_0(\lambda)\chi\|_{L^2\to L^2} \lesssim |\lambda|^{-1}\mathrm{e}^{L(\Im\lambda)_-} \tag{3.2}$$

with $L > \operatorname{diam}\operatorname{supp}\chi$, which proves the case for $s = t = 0$. For a given function $f \in H^1(\mathbb{R})$, we have

$$\begin{aligned} D_x\left(\chi\mathcal{R}_0\left(\lambda\right)\chi f\right) &= D_x\left(\chi\left(G_\lambda * \left(\chi f\right)\right)\right) = \left(D_x\chi\right)\left(G_\lambda * \left(\chi f\right)\right) + \chi\left(G_\lambda * D_x\left(\chi f\right)\right) \\ &= \left(D_x\chi\right)\left(G_\lambda * \left(\chi f\right)\right) + \chi\left(G_\lambda * D_x\chi f\right) + \chi\left(G_\lambda * D_x f\chi\right) \\ &= \left(D_x\chi\right)\left(\mathcal{R}_0\left(\lambda\right)\chi f\right) + \chi\mathcal{R}_0\left(\lambda\right)D_x\chi f + \chi\mathcal{R}_0\left(\lambda\right)\chi D_x f \\ &= \left(D_x\chi\right)\left(\chi_1\mathcal{R}_0\left(\lambda\right)\chi_1\chi f\right) + \chi\chi_1\mathcal{R}_0\left(\lambda\right)\chi_1 D_x\chi f + \chi\chi_1\mathcal{R}_0(\lambda)\chi_1\chi D_x f, \end{aligned} \tag{3.3}$$

where $D_x v := v'$, the function G_λ is given as $G_\lambda(x) = \frac{i}{2\lambda}\mathrm{e}^{\mathrm{i}\lambda|x|}$ and $\chi_1 \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $\chi_1|_{\mathrm{supp}\chi} = 1$ and $L > \mathrm{diam}\,\mathrm{supp}\chi_1$. Combining (3.2) and (3.3), we obtain

$$\begin{aligned} \|\chi\mathcal{R}_0\left(\lambda\right)\chi f\|_{H^1(\mathbb{R})} &\lesssim \|\chi\mathcal{R}_0\left(\lambda\right)\chi f\|_{L^2(\mathbb{R})} + \|D_x\left(\chi\mathcal{R}_0\left(\lambda\right)\chi f\right)\|_{L^2(\mathbb{R})} \\ &\lesssim \|\chi\mathcal{R}_0\left(\lambda\right)\chi f\|_{L^2(\mathbb{R})} + \|\chi_1\mathcal{R}_0\left(\lambda\right)\chi_1\chi f\|_{L^2(\mathbb{R})} + \|\chi_1\mathcal{R}_0\left(\lambda\right)\chi_1 D_x\chi f\|_{L^2(\mathbb{R})} \\ &\quad + \|\chi_1\mathcal{R}_0\left(\lambda\right)\chi\chi_1 D_x f\|_{L^2(\mathbb{R})} \\ &\lesssim |\lambda|^{-1}\mathrm{e}^{L(\Im\lambda)_-}\left(\|f\|_{L^2(\mathbb{R})} + \|\chi f\|_{L^2(\mathbb{R})} + \|D_x\chi f\|_{L^2(\mathbb{R})} + \|\chi D_x f\|_{L^2(\mathbb{R})}\right) \\ &\lesssim |\lambda|^{-1}\mathrm{e}^{L(\Im\lambda)_-}\|f\|_{H^1(\mathbb{R})}, \end{aligned}$$

which yields the resolvent estimate for $s = t = 1$ and then the estimates for all $s = t \in \mathbb{N}$ by induction. Using the interpolation of fractional order Sobolev spaces in [2, (7.57)], i.e.

$$H^s\left(\Omega\right) = \left[L^2\left(\Omega\right), H^n\left(\Omega\right)\right]_{s/n}, \quad n > s > 0, \quad n \in \mathbb{N},$$

we obtain the resolvent estimates for all $s = t \geqslant 0$.

For the case $s = t < 0$, by the definition of convolution of distributions and the symmetry of the fundamental solution, we have

$$\begin{aligned} \left\langle\chi\mathcal{R}_0\left(\lambda\right)\chi f, \phi\right\rangle &= \left\langle G_\lambda * \left(\chi f\right), \chi\phi\right\rangle = \left\langle\chi f, \left\langle G_\lambda\left(x-y\right), \phi\chi\right\rangle_x\right\rangle_y \\ &= \left\langle\left\langle\chi f, G_\lambda\left(y-x\right), \phi\chi\right\rangle_x\right\rangle_y \\ &= \left\langle\chi f, G_\lambda * \left(\phi\chi\right)\right\rangle = \left\langle f, \chi\mathcal{R}_0\left(\lambda\right)\chi\phi\right\rangle, \quad \forall\phi \in C_0^\infty\left(\mathbb{R}\right), \end{aligned}$$

which implies

$$\left|\left\langle\chi\mathcal{R}_0\left(\lambda\right)\chi f, \phi\right\rangle\right| \lesssim \|f\|_{H^s}\|\chi\mathcal{R}_0\left(\lambda\right)\chi\phi\|_{H^{-s}} \lesssim |\lambda|^{-1}\mathrm{e}^{L(\Im\lambda)_-}\|f\|_{H^s}\|\phi\|_{H^{-s}}.$$

Thus, we obtain that the resolvent $\chi R_0(\lambda)\chi : H^s \to H^s$ is a bounded operator and the inequality

$$\|\chi\mathcal{R}_0\left(\lambda\right)\chi\|_{H^s\to H^s} \lesssim |\lambda|^{-1}\mathrm{e}^{L(\Im\lambda)_-}$$

holds for all $s \in \mathbb{R}$. Then noting

$$D_x\left(\chi\mathcal{R}_0\left(\lambda\right)\chi f\right) = \chi'\mathcal{R}_0\left(\lambda\right)\chi f + \chi D_x\left(\mathcal{R}_0\left(\lambda\right)\chi f\right),$$

we have

$$\begin{aligned} D_x\left(\mathcal{R}_0\left(\lambda\right)\chi f\right)(x) &= \frac{i}{2\lambda}\int_{\mathbb{R}}\partial_x\left(\mathrm{e}^{\mathrm{i}k|x-y|}\right)\chi\left(y\right)f(y)\,\mathrm{d}y \\ &= -\frac{1}{2}\int_{\mathbb{R}}\frac{(x-y)}{|x-y|}\mathrm{e}^{\mathrm{i}k|x-y|}\chi\left(y\right)f(y)\,\mathrm{d}y, \end{aligned}$$

which implies

$$\|\chi\mathcal{R}_0\left(\lambda\right)\chi\|_{H^s\to H^{s+1}} \lesssim \left(|\lambda|^{-1}+1\right)\mathrm{e}^{L(\Im\lambda)_-}$$

by similar arguments as above. As the following identity holds

$$\begin{aligned} D_x^2\left(\chi R_0(\lambda)\chi\right) &= \chi D_x^2 \mathcal{R}_0(\lambda)\chi + \left[D_x^2,\chi\right]\mathcal{R}_0(\lambda)\chi \\ &= \left(-1-\lambda^2\right)\chi\mathcal{R}_0(\lambda)\chi + \left[D_x^2,\chi\right]\mathcal{R}_0(\lambda)\chi, \end{aligned}$$

where $[D_x^2,\chi] := D_x^2\chi - \chi D_x^2$ is a first order differential operator, the corresponding estimates for $t=s$ and $t=s+1$ imply the resolvent estimate for $t=s+2$. The remaining cases can be obtained by applying the interpolation inequality.

Next, we show the analyticity. Fixing $s\in\mathbb{R}$ and $t\in[s,s+2]$, for any $f\in H^s$ and $g\in H^{-t}$, there exist sequences $\{f_n\},\{g_n\}\subset C_0^\infty(\mathbb{R})$ such that $f_n\to f$ in H^s and $g_n\to g$ in H^{-t}. Noting that

$$\langle\chi\mathcal{R}_0(\lambda)\chi f_n, g_n\rangle = \frac{i}{2\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi(x)\,\mathrm{e}^{\mathrm{i}\lambda|x-y|}\chi(y) f_n(y)\,g_n(x)\,\mathrm{d}x\mathrm{d}y,$$

we have

$$\begin{aligned} &\partial_\lambda\int_{\mathbb{R}}\int_{\mathbb{R}}\chi(x)\,\mathrm{e}^{\mathrm{i}\lambda|x-y|}\chi(y) f_n(y)\,g_n(x)\,\mathrm{d}x\mathrm{d}y \\ &\quad = i\int_{\mathbb{R}}\int_{\mathbb{R}}\chi(x)\,|x-y|\mathrm{e}^{\mathrm{i}\lambda|x-y|}\chi(y) f_n(y)\,g_n(x)\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

by the dominated convergence theorem, which yields that $\langle\chi\mathcal{R}_0(\lambda)\chi f_n,g_n\rangle$ is analytic for $\lambda\in\mathbb{C}\backslash\{0\}$. For any $K\subset\subset\mathbb{C}\backslash\{0\}$, by the resolvent estimate derived above, we have $\langle\chi\mathcal{R}_0(\lambda)\chi f_n,g_n\rangle\to\langle\chi\mathcal{R}_0(\lambda)\chi f,g\rangle$ uniformly for $\lambda\in K$. Hence, $\langle\chi\mathcal{R}_0(\lambda)\chi f,g\rangle$ is analytic for $\lambda\in\mathbb{C}\backslash\{0\}$. As a result, $\chi\mathcal{R}_0(\lambda)\chi : H^s\to H^t$ is a family of analytic operators for $\lambda\in\mathbb{C}\backslash\{0\}$.

□

The following lemma concerns the multiplication operator by the rough potential q.

Lemma 3.2. *Assume that $q\in W^{s,p}_{\mathrm{comp}}(\mathbb{R})$ with $s\in[-\frac{1}{2},0)$ and $p\in[-\frac{1}{s},+\infty)$. We have*

$$\|qf\|_{H^s(\mathbb{R})}\lesssim\|q\|_{W^{s,p}(\mathbb{R})}\|\chi f\|_{H^{-s}(\mathbb{R})},\quad \forall f\in H^{-s}_{\mathrm{loc}}(\mathbb{R}),$$

where $\chi\in C_0^\infty(\mathbb{R}^3)$ is a cut-off function with $\chi|_{\mathrm{supp}\,q}=1$.

Denote the resolvent of the Helmholtz equation by $R_q(\lambda)=(-D_x^2-\lambda^2-q)^{-1}$. The following lemma is concerned with the meromorphic continuation and resolvent estimates for $\mathcal{R}_q(\lambda)$.

Lemma 3.3. *Assume $q\in W^{s,p}_{\mathrm{comp}}(\mathbb{R})$ with $s\in[-\frac{1}{2},0)$ and $p\in[-\frac{1}{s},+\infty)$. Then the resolvent $\mathcal{R}_q(\lambda)$ is meromorphic for $\lambda\in\mathbb{C}$ with $\Im\lambda>0$ as a family of operators $\mathcal{R}_q(\lambda):H^s\to H^t$ with $t\in[s,s+2]$, which can be extended to a family of meromorphic operators as*

$$\mathcal{R}_q(\lambda): H^s_{\mathrm{comp}}\to H^t_{\mathrm{loc}},\quad \lambda\in\mathbb{C}$$

with $t\in[s,s+2]$. Further, fix a cut-off function $\chi\in C_0^\infty(\mathbb{R})$. Then the resolvent $\chi\mathcal{R}_q(\lambda)\chi: H^s\to H^t$ is analytic for $\lambda\in\mathscr{S}$ where

$$\mathscr{S} := \{\lambda\in\mathbb{C}:\Im\lambda\geqslant -A-\delta\log(1+|\lambda|),|\lambda|>C_0\}.$$

Moreover, there exist constants A,C_0,δ,C and L such that the resolvent estimates

$$\|\chi\mathcal{R}_q(\lambda)\chi\|_{H^s\to H^t}\leqslant C|\lambda|^{-1}(1+|\lambda|)^{t-s}\,\mathrm{e}^{L(\Im\lambda)_-}$$

hold for $t \in [s, s+2]$ *and* $\lambda \in \mathscr{S}$. *The constants* A, δ, L *depend on* χ *and* $\operatorname{supp} q$. *The constants* C, C_0 *depend on* χ, $\|q\|_{W^{s,p}(\mathbb{R})}$.

Proof. There formally holds the resolvent identity

$$\mathcal{R}_q(\lambda) = \mathcal{R}_0(\lambda)\left(I + qR_0(\lambda)\right)^{-1}.$$

Similar as the proof in lemma 3.1, by lemma 3.2 we obtain

$$\|q\mathcal{R}_0(\lambda)\|_{\mathcal{L}(H^s,H^s)} \lesssim \|q\|_{W^{s,p}} \|\mathcal{R}_0(\lambda)\|_{\mathcal{L}(H^s,H^{-s})} \lesssim |\lambda|^{-1}(1+|\lambda|)^{-2s}$$

for $\Im\lambda > 0$. Then when $\Im\lambda > 0$ and $|\lambda| >> 1$, the operator $I + q\mathcal{R}_0(\lambda) : H^s \to H^s$ is invertible and

$$\left(I + q\mathcal{R}_0(\lambda)\right)^{-1} = \sum_{j=0}^{\infty} \left(-q\mathcal{R}_0(\lambda)\right)^j.$$

Using lemma 3.1 and analytic Fredholm theory [10, theorem C.8], the following family of operators

$$\mathcal{R}_q(\lambda) = \mathcal{R}_0(\lambda)\left(I + qR_0(\lambda)\right)^{-1} : H^s \to H^t, \quad \Im\lambda > 0,$$

is meromorphic with respective to λ for $t \in [s, s+2]$. Next, we consider the operator $\chi R_0(\lambda)\chi$. There holds the resolvent identity

$$\chi\mathcal{R}_q(\lambda)\chi = \chi\mathcal{R}_0(\lambda)\chi_1\left(I + q\mathcal{R}_0(\lambda)\chi_1\right)^{-1}\left(1 - q\mathcal{R}_0(\lambda)(1-\chi_1)\right)\chi, \quad (3.4)$$

where $\chi_1 \in C_0^{\infty}(\mathbb{R}^3)$ such that $\chi_1 = 1$ on $\operatorname{supp} q$. From lemmas 3.1 and 3.2, we have

$$\|q\mathcal{R}_0(\lambda)\chi_1\|_{\mathcal{L}(H^s,H^s)} \lesssim \|q\|_{W^{s,p}(\mathbb{R}^3)} |\lambda|^{-1}(1+|\lambda|)^{-2s} \mathrm{e}^{L(\Im\lambda)_-}, \quad \lambda \in \mathscr{S}.$$

We see that for δ sufficiently small, say $\delta < \frac{1}{2L}$, and C_0 sufficiently large,

$$\|q\mathcal{R}_0(\lambda)\chi\|_{\mathcal{L}(H^s,H^s)} \lesssim |\lambda|^{-1/2} < 1.$$

Thus, from a Neumann series argument we have $I + q\mathcal{R}_0(\lambda)\chi : H^s_{\mathrm{comp}} \to H^s_{\mathrm{comp}}$ is invertible. From the resolvent identity (3.4), the free resolvent estimates in lemma 3.1 and the analytic Fredholm theory, we have that $\chi\mathcal{R}_q(\lambda)\chi : H^s \to H^t$ is analytic for $\lambda \in \mathscr{S}$ with the resolvent estimates

$$\|\chi\mathcal{R}_q(\lambda)\chi\|_{\mathcal{L}(H^s,H^t)} \lesssim |\lambda|^{-1}(1+|\lambda|)^{t-s} \mathrm{e}^{L(\Im\lambda)_-}, \quad t \in [s, s+2].$$

The proof is complete. □

Theorem 3.1. *Assume that the random potential* q *is a GMIG random field of order* $-m$ *in* $(-1,1)$ *with* $m > 0$. *For* k *sufficiently large, the scattering problem almost surely admits a unique solution* $u^{sc} \in H^{s+2}_{\mathrm{loc}}$ *with non-positive constant* s *satisfying* $-1/2 < s < (m-1)/2$. *Furthermore,* u^{sc} *can be represented by*

$$u^{sc} = \sum_{j=1}^{\infty} \left(\mathcal{R}_0(k)q\right)^j \left(u^{inc}\right).$$

Proof. From lemmas 2.1, 3.1 and 3.2, we have

$$\|\chi \mathcal{R}_0(k) q\|_{H^{-s}\to H^{-s}} \lesssim |k|^{-1}(1+|k|)^{-2s},$$

where $\chi \in C_0^\infty(\mathbb{R})$ and the non-positive constant s satisfies $-1/2 < s < (m-1)/2$. Fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ in supp$q$. Then we have the Neumann series expansion by the resolvent estimate with k being sufficiently large

$$u^{sc} := \sum_{j=1}^{\infty} (\mathcal{R}_0(k) q)^j (u^{inc}) = \mathcal{R}_0(k) q \left(\sum_{j=0}^{\infty} (\chi \mathcal{R}_0(k) q)^j (u^{inc}) \right).$$

Moreover, the series $\sum_{j=0}^{\infty} (\chi \mathcal{R}_0(k) q)^j (u^{inc})$ is convergent in H^{-s}. Then by applying lemmas 3.1 and 3.2 we obtain $u^{sc} \in H^{s+2}_{\mathrm{loc}}$, which is a solution to the scattering problem in the sense of distribution.

Next, we prove the uniqueness. It suffices to show the scattering problem

$$\begin{cases} (u^{sc})'' + k^2 u^{sc} + q u^{sc} = 0, & \text{in} \quad \mathbb{R} \\ u^{sc} = \begin{cases} a^+ e^{ikx}, & x \geqslant 1, \\ a^- e^{-ikx}, & x \leqslant -1, \end{cases} \end{cases}$$

only admits a trivial solution $u^{sc} = 0$. From [10, theorem 3.33], we know that if the scattering problem admits a non-trivial solution, then k must be the pole of the resolvent $\mathcal{R}_V(\lambda)$. On the other hand, lemma 3.3 shows that k is not a pole if it is sufficiently large, which implies the uniqueness. The proof is complete. □

4. The inverse random potential problem

In this section, we investigate the inverse scattering problem of determining the strength μ from boundary measurements.

Rewrite the Helmholtz equation as follows

$$(u^{sc})'' + k^2 u^{sc} + q u^{sc} = -q e^{ikx}. \tag{4.1}$$

Multiplying both sides of the equation (4.1) by e^{ikx}, using integration by parts and the radiation condition (1.2), we obtain

$$\begin{aligned} -\widehat{q}(-2k) = -\int_{\mathbb{R}} q(x) e^{2ikx} \mathrm{d}x &= \int_{\mathbb{R}} q(x) u^{sc}(x) e^{ikx} \mathrm{d}x + (u^{sc})' e^{ikx}|_{-1}^{1} - iku^{sc} e^{ikx}|_{-1}^{1} \\ &= \int_{\mathbb{R}} q(x) u^{sc}(x) e^{ikx} \mathrm{d}x - 2iku^{sc}(-1) e^{-ik}. \end{aligned} \tag{4.2}$$

On the other hand, the Fourier transform of q is related to the micro-correlation strength of q as follows

$$\begin{aligned}\mathbb{E}\left(\overline{\hat{q}(-2(k+\tau))}\hat{q}(-2k)\right) &= \mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} q(x)\,q(y)\,\mathrm{e}^{-\mathrm{i}\,2(k+\tau)x}\mathrm{e}^{2\mathrm{i}ky}\,\mathrm{d}x\mathrm{d}y\right)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_q(x,y)\,\mathrm{e}^{-\mathrm{i}\,2(k+\tau)x}\mathrm{e}^{2\mathrm{i}ky}\,\mathrm{d}x\mathrm{d}y\\ &= (2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\,(2(k+\tau)x-2ky)}\mathrm{e}^{\mathrm{i}(x-y)\cdot\zeta}c_q(x,\zeta)\,\mathrm{d}\zeta\mathrm{d}x\mathrm{d}y\\ &= \int_{-1}^{1}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x(\zeta-2(k+\tau))}c_q(x,\zeta)\,\delta(2k-\zeta)\,\mathrm{d}\zeta\mathrm{d}x\\ &= \int_{-1}^{1} \mathrm{e}^{-2\mathrm{i}x\tau}c(x,2k)\,\mathrm{d}x\\ &= \int_{-1}^{1} \mathrm{e}^{-2\mathrm{i}x\tau}\mu(x)\,|2k|^{-m}\,\mathrm{d}x+\mathcal{O}\left(|2k|^{-(m+1)}\right)\\ &= 2^{-m}k^{-m}\hat{\mu}(2\tau)+\mathcal{O}(k^{-m-1}).\end{aligned} \tag{4.3}$$

Moreover, by an argument of ergodicity in [13] we obtain the following limit.

$$\lim_{K\to\infty}\frac{1}{K}\int_{K}^{2K} k^m\left(\overline{q(-2(k+\tau))}\widehat{q}(-2k)-\mathbb{E}\overline{q(-2(k+\tau))}\widehat{q}(-2k)\right)\mathrm{d}k=0. \tag{4.4}$$

For the reader's convenience, we next give a sketch of the proof of (4.4). The argument of ergodicity is based on the following lemma [8, p 91]

Lemma 4.1. *Let X_{t} with $t\geqslant 0$ be a real-valued stochastic process with continuous paths. In addition, the process X_{t} satisfies $\mathbb{E}X_t=0$ and there exist non-negative constants c_0 c_1, c_2, satisfying $0\leqslant 2c_1<c_2$ such that*

$$|\mathbb{E}(X_sX_t)|\lesssim\frac{s^{c_1}+t^{c_1}}{1+|s-t|^{c_2}}$$

for any $s,t\geqslant c_0$. Then we almost surely have

$$\lim_{T\to\infty}\frac{1}{T}\int_{T}^{2T}X_t\,\mathrm{d}t=0.$$

Denote $Y_k=\widehat{q}(-2k)$. A direct calculation gives

$$\overline{\widehat{q}(-2(k+\tau))}\widehat{q}(-2k)-\mathbb{E}\left[\overline{\widehat{q}(-2(k+\tau))}\widehat{q}(-2k)\right]=\sum_{W_k\in\mathcal{A}}C(W_k)\left(W_k^2-\mathbb{E}W_k^2\right), \tag{4.5}$$

where

$$\begin{aligned}\mathcal{A}:=\{&\Re Y_k,\Im Y_k,\Re Y_{k+\tau},\Im Y_{k+\tau},\Re Y_k-\Re Y_{k+\tau},\\ &\Im Y_k-\Im Y_{k+\tau},\Re Y_{k+\tau}+\Im Y_k,\Im Y_{k+\tau}-\Re Y_k\}\end{aligned}$$

and the constant $C(W_k)\in\{\frac{1-i}{2},-\frac{1}{2},\frac{i}{2}\}$. Let $X_k:=k^m(W_k^2-\mathbb{E}W_k^2)$. According to [13, lemma 3.4], there holds

$$\mathbb{E}(X_{k_1}X_{k_2})\lesssim(1+|k_1-k_2|)^{-N}$$

for all $N>0$. Hence, we are able to apply lemma 4.1 to obtain that

$$\lim_{K\to\infty}\frac{1}{K}\int_{K}^{2K}X_k\,\mathrm{d}k=0. \tag{4.6}$$

Combining (4.5) and (4.6) we arrive at (4.4).

Combining (4.3) and (4.4), we have the following result.

Lemma 4.2. *There almost surely holds the limit*

$$\widehat{\mu}(2\tau) = 2^m \lim_{K\to\infty} \frac{1}{K}\int_K^{2K} k^m \overline{\widehat{q}(-2(k+\tau))}\widehat{q}(-2k)\,\mathrm{d}k.$$

Based on the integral identity (4.2), we introduce the boundary measurement at $x=-1$

$$\mathcal{E}(k) := -2iku^{sc}(-1,k)\,\mathrm{e}^{-\mathrm{i}k}.$$

The notation $u^{sc}(-1,k)$ exhibits the dependence of the scattered field at $x=-1$ on the wavenumber k.

According to lemma 4.2, to derive the reconstruction formula, we shall estimate the error between the correlation data $\mathcal{E}(k+\tau)\overline{\mathcal{E}(k)}$ and $\widehat{q}(2(k+\tau))\overline{\widehat{q}(2k)}$. Recalling the series expansion

$$\begin{aligned} u^{sc} &= \sum_{j=0}^{\infty} (\mathcal{R}_0(k)\,q)^j \left(\mathcal{R}_0(k)\left(qu^{inc}\right)\right) \\ &= \mathcal{R}_0(k)\left(qu^{inc}\right) + \sum_{j\geqslant 2}^{+\infty} (\mathcal{R}_0(k)\,q)^j u^{inc} \\ &:= u_1 + u_2, \end{aligned}$$

which gives

$$\mathcal{E}(k) = -\widehat{q}(-2k) - \int_{\mathbb{R}} q(x)\,u_1(x,k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x - \int_{\mathbb{R}} q(x)\,u_2(x,k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{4.7}$$

by (4.2). We obtain the following error estimate by (4.7) and Cauchy–Schwarz inequality

$$\begin{aligned} &\frac{1}{K}\int_K^{2K} k^m \left|\overline{\mathcal{E}(k+\tau)}\mathcal{E}(k) - \overline{\widehat{q}(-2(k+\tau))}\widehat{q}(-2k)\right|\mathrm{d}k \\ &\quad\leqslant \sum_{j,l=1}^{2} \frac{1}{K}\left(\int_K^{2K} k^m |I_j(k+\tau)|^2 + |I_l(k)|^2\,\mathrm{d}k\right) \\ &\qquad + \sum_{j=1}^{2}\sqrt{\frac{1}{K}\int_K^{2K} k^m |I_j(k+\tau)|^2\,\mathrm{d}k}\sqrt{\frac{1}{K}\int_K^{2K} k^m |\widehat{q}(-2k)|^2\,\mathrm{d}k} \\ &\qquad + \sum_{l=1}^{2}\sqrt{\frac{1}{K}\int_K^{2K} k^m |I_l(k)|^2\,\mathrm{d}k}\sqrt{\frac{1}{K}\int_K^{2K} k^m |\widehat{q}(-2(k+\tau))|^2\,\mathrm{d}k}, \end{aligned}$$

where $I_j(k) := \int_{\mathbb{R}} q(x)u_j(x,k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$. Consequently, it is sufficient to prove that

$$\lim_{K\to\infty}\int_K^{2K} k^m |I_j(k)|^2\,\mathrm{d}k = 0, \quad j=1,2. \tag{4.8}$$

We first estimate $I_1(k)$. To this end, we adopt some ideas from [6]. From a direct calculation, we have

$$\begin{aligned} I_1(k) = \int_{\mathbb{R}} q(x)\, u_1(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x &= \frac{i}{2k} \int_{\mathbb{R}} \int_{\mathbb{R}} q(x)\, q(y)\, \mathrm{e}^{\mathrm{i}k((x+y)+|x-y|)}\, \mathrm{d}y\, \mathrm{d}x \\ &= \frac{i}{2k} \int_{\mathbb{R}} \int_{\mathbb{R}} q\left(\frac{z+z'}{2}\right) q\left(\frac{z-z'}{2}\right) \mathrm{e}^{\mathrm{i}k\left(z+|z'|\right)}\, \mathrm{d}z\, \mathrm{d}z' \\ &= \frac{i}{2k} \int_0^{+\infty} \int_{\mathbb{R}} q\left(\frac{z+z'}{2}\right) q\left(\frac{z-z'}{2}\right) \mathrm{e}^{\mathrm{i}k\left(z+z'\right)}\, \mathrm{d}z\, \mathrm{d}z' \\ &\quad + \frac{i}{2k} \int_{-\infty}^0 \int_{\mathbb{R}} q\left(\frac{z+z'}{2}\right) q\left(\frac{z-z'}{2}\right) \mathrm{e}^{\mathrm{i}k\left(z-z'\right)}\, \mathrm{d}z\, \mathrm{d}z' \\ &= \frac{i}{k} \int_0^{\infty} \int_{\mathbb{R}} q\left(\frac{z+z'}{2}\right) q\left(\frac{z-z'}{2}\right) \mathrm{e}^{\mathrm{i}k\left(z+z'\right)}\, \mathrm{d}z\, \mathrm{d}z' \\ &= \frac{i}{k} \int_{\mathbb{R}} \int_{\mathbb{R}} J(z,z')\, \mathrm{e}^{\mathrm{i}k\left(z+z'\right)}\, \mathrm{d}z\, \mathrm{d}z', \end{aligned}$$

where

$$J(z,z') := 1_{[0,\infty]}(z')\, q\left(\frac{z+z'}{2}\right) q\left(\frac{z-z'}{2}\right).$$

We next introduce the mollification $q_\epsilon := q * \chi_\epsilon$, where $\chi_\epsilon(x) = \frac{1}{\epsilon}\chi(\frac{x}{\epsilon})$ with $\chi \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \chi\, \mathrm{d}x = 1$. Then denote

$$I_{1,\epsilon}(k) = \frac{i}{k} \int_{\mathbb{R}} \int_{\mathbb{R}} J_\epsilon(z,z')\, \mathrm{e}^{\mathrm{i}k\left(z'+z\right)}\, \mathrm{d}z\, \mathrm{d}z'$$

with

$$J_\epsilon(z,z') := 1_{[0,\infty]}(z')\, q_\epsilon\left(\frac{z+z'}{2}\right) q_\epsilon\left(\frac{z-z'}{2}\right).$$

By the Fatou's lemma, the inequality

$$\limsup_{\epsilon\to 0} \mathbb{E} \int_1^\infty k^{m-1} |I_{1,\epsilon}(k)|^2\, \mathrm{d}k < \infty \tag{4.9}$$

implies that

$$\int_1^\infty k^{m-1} |I_1(k)|^2\, \mathrm{d}k < \infty$$

almost surely holds, which ensures (4.8) for $j=1$ by the dominated convergence theorem. Thus, we turn to prove (4.9). There exists $R>0$ such that

$$\begin{aligned} \int_1^\infty k^{m-1} |I_{1,\epsilon}(k)|^2\, \mathrm{d}k &\leqslant \int_1^\infty k^{m-3} \left| \int_{|z|<R} \mathrm{e}^{\mathrm{i}kz} \int_{|z'|<R} \mathrm{e}^{\mathrm{i}kz'} J_\epsilon(z,z')\, \mathrm{d}z'\mathrm{d}z \right|^2 \mathrm{d}k \\ &\lesssim \int_1^\infty k^{m-3} \int_{|z|<R} \left| \int_{|z'|<R} \mathrm{e}^{\mathrm{i}kz'} J_\epsilon(z,z')\, \mathrm{d}z' \right|^2 \mathrm{d}z\, \mathrm{d}k. \end{aligned}$$

By taking expectation, we have

$$\begin{aligned}&\limsup_{\epsilon\to 0}\mathbb{E}\int_1^\infty k^{m-1}|I_{1,\epsilon}(k)|^2\,\mathrm{d}k\\&\quad\leqslant\limsup_{\epsilon\to 0}\int_1^\infty k^{m-3}\int_{|z|<R}\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\mathrm{d}z\mathrm{d}k.\end{aligned}\tag{4.10}$$

In order to show the right-hand side of (4.10) is finite, it is sufficient to prove

$$\limsup_{\epsilon\to 0}\int_{|z|<R}\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\mathrm{d}z\lesssim\begin{cases}1, & 1/2<m<2,\\ k^{-2}, & 2\leqslant m<4.\end{cases}\tag{4.11}$$

To this end, applying the Isserli's theorem yields

$$\begin{aligned}&\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\\&=\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\mathbb{E}\left(q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\right)\mathbb{E}\left(q_\epsilon\left(\frac{z+\tilde{z}}{2}\right)q_\epsilon\left(\frac{z-\tilde{z}}{2}\right)\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'\\&\quad+\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\mathbb{E}\left(q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-\tilde{z}}{2}\right)\right)\mathbb{E}\left(q_\epsilon\left(\frac{z+\tilde{z}}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'\\&\quad+\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\mathbb{E}\left(q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z+\tilde{z}'}{2}\right)\right)\mathbb{E}\left(q_\epsilon\left(\frac{z-\tilde{z}}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'.\end{aligned}\tag{4.12}$$

Moreover, it is straightforward to verify that

$$\lim_{\epsilon\to 0}\mathbb{E}\left[q_\epsilon(x)\,q_\epsilon(y)\right]=K_q(x,y)$$

in the sense of distribution. As $K_q(x,y)$ is weakly singular when $m>0$, the above limit holds point-wisely. As a consequence, by taking the limit in (4.12), we derive

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_{|z|<R}\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\mathrm{d}z\\&\quad\leqslant\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}K_q\left(\frac{z+z'}{2},\frac{z-z'}{2}\right)K_q\left(\frac{z+\tilde{z}}{2},\frac{z-\tilde{z}}{2}\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'\,\mathrm{d}z\\&\qquad+\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}K_q\left(\frac{z+z'}{2},\frac{z-\tilde{z}}{2}\right)K_q\left(\frac{z+\tilde{z}}{2},\frac{z-z'}{2}\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'\,\mathrm{d}z\\&\qquad+\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}K_q\left(\frac{z+z'}{2},\frac{z+\tilde{z}}{2}\right)K_q\left(\frac{z-\tilde{z}}{2},\frac{z-z'}{2}\right)\mathrm{d}\tilde{z}\,\mathrm{d}z'\,\mathrm{d}z.\end{aligned}$$

Now we consider the following two cases. When $1/2<m<2$ and $m\neq 1$, from lemma 2.2, we obtain

$$\limsup_{\epsilon\to 0}\int_{|z|<R}\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\mathrm{d}z$$
$$\lesssim\int_{|z|<R}\int_0^R\int_0^R |z'|^{m-1}|\tilde{z}|^{m-1}+|z'+\tilde{z}|^{2(m-1)}+|z'-\tilde{z}|^{2(m-1)}\,\mathrm{d}\tilde{z}\mathrm{d}z'\mathrm{d}z<\infty.$$

As $m=1$, the above estimate similarly holds. When $2\leqslant m<4$, by using integration by parts we have

$$\begin{aligned}
&\limsup_{\epsilon\to 0}\int_{|z|<R}\mathbb{E}\left|\int_0^R \mathrm{e}^{\mathrm{i}kz'}q_\epsilon\left(\frac{z+z'}{2}\right)q_\epsilon\left(\frac{z-z'}{2}\right)\mathrm{d}z'\right|^2\mathrm{d}z\\
&\quad\leqslant\frac{1}{k^2}\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\partial_{z'}K_q\left(\frac{z+z'}{2},\frac{z-z'}{2}\right)\partial_{\tilde{z}}K_q\left(\frac{z+\tilde{z}}{2},\frac{z-\tilde{z}}{2}\right)\mathrm{d}\tilde{z}\mathrm{d}z'\mathrm{d}z\\
&\qquad+\frac{1}{k^2}\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\partial_{z'}\partial_{\tilde{z}}\left(K_q\left(\frac{z+z'}{2},\frac{z-\tilde{z}}{2}\right)K_q\left(\frac{z+\tilde{z}}{2},\frac{z-z'}{2}\right)\right)\mathrm{d}\tilde{z}\mathrm{d}z'\mathrm{d}z\\
&\qquad+\frac{1}{k^2}\int_{|z|<R}\int_0^R\int_0^R \mathrm{e}^{\mathrm{i}k(z'-\tilde{z})}\partial_{\tilde{z}}\partial_{z'}\left(K_q\left(\frac{z+\tilde{z}}{2},\frac{z+z'}{2}\right)K_q\left(\frac{z-\tilde{z}}{2},\frac{z-z'}{2}\right)\right)\mathrm{d}\tilde{z}\mathrm{d}z'\mathrm{d}z\\
&\quad\lesssim\frac{1}{k^2}\int_{|z|<R}\int_0^R\int_0^R |z'|^{m-2}|\tilde{z}|^{m-2}+|z'+\tilde{z}|^{2(m-2)}+|z'-\tilde{z}|^{2(m-2)}\,\mathrm{d}\tilde{z}\mathrm{d}z'\mathrm{d}z\\
&\quad\lesssim\frac{1}{k^2},
\end{aligned}$$

which completes the proof of (4.11). Thus, we show that the right-hand side of (4.10) is finite, which gives (4.9). As a result, we arrive at the following lemma.

Lemma 4.3. *When $1/2\leqslant m<4$, there almost surely holds*

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}k^m|I_1(k)|^2\,\mathrm{d}k=0.$$

The higher-order terms can be estimated by resolvent estimates directly as follows

$$|I_2(k)|\leqslant\sum_{j\geqslant 2}\|\mathrm{e}^{\mathrm{i}kx}q\|_{H^\alpha}\|\chi\left(\mathcal{R}_0(k)q\right)^j\mathrm{e}^{\mathrm{i}kx}\|_{H^{-\alpha}}\lesssim\sum_{j\geqslant 2}\|\mathrm{e}^{\mathrm{i}kx}\chi\|_{H^\alpha}^2k^{-j(1+2\alpha)}\lesssim k^{-2-6\alpha},$$

where α is a non-positive constant satisfying $\alpha<(m-1)/2$ and $\chi\in C_0^\infty(\mathbb{R})$ is a cut-off function satisfying $\chi=1$ in $\operatorname{supp}q$. Hence, we have the following estimate.

Lemma 4.4. *When $2/5<m<4$, there almost surely holds*

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}k^m|I_2(k)|^2\,\mathrm{d}k=0.$$

Summing up the above analysis, we obtain the main result in this paper.

Theorem 4.1. *Suppose that q is a GMIG random field in $(-1,1)$ of order $-m$ with $m\in(1/2,4)$. Then it almost surely holds*

$$\widehat{\mu}(2\tau)=2^m\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}k^m\overline{\mathcal{E}(k+\tau)}\mathcal{E}(k)\,\mathrm{d}k$$

for any $\tau \in \mathbb{R}$, *where the boundary measurement* $\mathcal{E}(k)$ *is given by*

$$\mathcal{E}(k) := -2iku^{sc}(-1,k)\,\mathrm{e}^{-\mathrm{i}k} = -2ika^{-}(k).$$

The above theorem shows that the micro-correlation strength can be reconstructed by a single realization of the data at a single point averaged over the high-frequency band almost surely. As a consequence, by using analyticity of the resolvent in lemma 3.3 and the analytic continuation in complex theory, we have that μ can be uniquely determined from the data measured at multiple frequencies only in a finite frequency band. The result is stated as follows.

Theorem 4.2. *Let* $1/2 < m < 4$. *Then* μ *can be uniquely determined from the data* $\{u^{sc}(-1,k) : k \in I\}$ *where* $I = [K_0, K_1]$ *with* $K_1 > K_0 > C_0$. *Here* C_0 *is specified in lemma* 3.3.

Remark 4.1. (1) The arguments for the one-dimensional case in this paper can also be extended to higher-dimensional case. Though uniqueness of determining the micro-correlation strength of the random potential has been obtained in [6, 12], it is worth noticing that in previous works, the analysis requires the data to be measured in a bounded domain. In contrast, the method developed in this paper only requires the boundary measurements.

(2) In theorems 4.1 and 4.2, the micro-correlation strength μ is uniquely determined from the data at the point $x = -1$. If the incident plane wave $\mathrm{e}^{\mathrm{i}kx}$ is replaced by $\mathrm{e}^{-\mathrm{i}kx}$, then μ can be recovered from the data at the point $x = 1$.

5. Inverse random source problem

In this section, we consider an inverse random source scattering problem. The random source scattering problem is formulated as

$$\begin{cases} u'' + k^2 u = f, & \text{in} \quad \mathbb{R}, \\ u = \begin{cases} a^{+}\mathrm{e}^{\mathrm{i}kx}, & x \geqslant 1, \\ a^{-}\mathrm{e}^{-\mathrm{i}kx}, & x \leqslant -1, \end{cases} \end{cases} \tag{5.1}$$

where f is a GMIG random field of order $-m$ in $(-1,1)$ with the micro-correlation strength μ. An inverse random source problem was studied in [15] for the following Helmholtz equation with a nonzero constant damping coefficient b

$$u'' + k^2 u + ikbu = f.$$

It was proposed in [15] that the inverse random source problem for the Helmholtz equation (5.1) without damping is open.

As an application of our new method, the above open problem can be solved. Specifically, similar to (4.3), we have the relation

$$\mathbb{E}\left[\widehat{f}((k+\tau))\overline{\widehat{f}(k)}\right] = k^{-m}\widehat{\mu}(\tau) + \mathcal{O}\left(k^{-1-m}\right).$$

Multiplying the equation by $\mathrm{e}^{\mathrm{i}kx}$ and using integration by parts yield

$$\widehat{f}(-k) = \int_{\mathbb{R}} f(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = -2iku(-1,k)\,\mathrm{e}^{-\mathrm{i}k} := \mathcal{E}(k).$$

Repeating the arguments for the inverse potential problem, the strength of the random source without damping can be uniquely reconstructed as follows.

Theorem 5.1. *The micro-correlation strength of the random source can be uniquely reconstructed by*

$$\widehat{\mu}(\tau) = \lim_{K\to\infty} \frac{1}{K} \int_K^{2K} k^m \mathcal{E}(k+\tau)\,\overline{\mathcal{E}(k)}\,\mathrm{d}k,$$

which almost surely holds for any $\tau \in \mathbb{R}$.

6. Conclusion

In this paper, we have discussed the direct and inverse random potential scattering problems in one dimension. For the direct problem, an analytic domain and resolvent estimates are obtained for the resolvent of the elliptic operator with rough potentials by scattering theory. For the inverse problem, by using correlation Dirichlet measurements at a single point averaged over high-frequency band generated from a single realization, a reconstruction formula for the principal symbol of the random potential is derived almost surely. The method developed in this work is unified and can be applied to inverse random source problems and stochastic inverse problems in higher dimensions. A possible continuation of this work is to investigate the stability of the inverse random scattering problem, which is more challenging. We hope to report the progress on this subject elsewhere in the future.

Data availability statement

No new data were created or analysed in this study.

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